Optimization problem of portfolios with an illiquid asset

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Ein Optimierungsproblem für Portfolios mit einer illiquiden Position

Abstract

This work is devoted to the problem of liquidity that draws a lot of attention after the global financial crisis. We consider an optimization problem for a portfolio with an illiquid, a risky and a riskless liquid assets. We work in Merton's optimal consumption framework with continuous time. The liquid part of the investment is described by a standard Black-Scholes market. The illiquid asset is sold at an exogenous random moment with prescribed distribution and generates additional liquid wealth dependent on its paper value. We show that one can consider a problem with infinite time horizon and special weight function that is characterized by the probability distribution of the liquidation time instead of a problem with an exogenous random liquidation time. Using the viscosity solution techniques, developed for the problem of optimization in presence of a random income, we prove the existence and uniqueness of the solution for the considered problem with logarithmic utility and modest restrictions on the liquidation time distribution. We find asymptotic bounds for the value function when liquidation time has exponential or Weibull distribution. We find optimal policies in a feedback form and illustrate how they differ from classical Merton's policies. Through a Lie group analysis we find the admitted Lie algebra for a problem with general liquidation time distribution in cases of HARA and log utility functions and formulate corresponding theorems for all these cases. Using these Lie algebras we obtain reduced equations of the lower dimension for the studied three dimensional partial differential equations. Several of similar substitutions were used in other works before, whereas others are new to our knowledge. The applied method of Lie group analysis gives us the possibility to provide a complete set of non-equivalent substitutions and reduced equations that was not provided for the problem of such type so far. Further research of these equations with numerical and quantitative methods is expected to benefit from such analysis.

Zusammenfassung

Diese Arbeit ist dem in der globalem Finanzkrise hochaktuellem Problem der Liquidität gewidmet. Wir betrachten ein Optimierungsproblem für ein Portfolio mit einer illiquiden, und einer liquiden risikoreichen und risikolosen Position.

Wir arbeiten im Rahmen des Mertonschen optimalen Verbrauchsproblems mit stetiger Zeit. Der liquide Anteil der Investitionen wird mit dem klassischen Black-Scholes Marktmodell beschrieben. Die illiquide Position wird an einem exogenen zufäligen Zeitpunkt mit einer vorgeschriebenen Verteilung verkauft und erzeugt ein zusätzliches liquides Vermögen welches von seinem Bilanzierungswert abhängig ist.

Wir zeigen, dass man ein Problem mit einem unendlichen Zeithorizont und einer speziellen Gewichtsfunktion, die durch die Wahrscheinlichkeitsverteilung der Liquidationszeit T bestimmt wird, anstelle des Problems mit einer exogenen zufäligen Liquidationszeit betrachten kann. Unter Ausnutzung der Methode der viskosen Lösungen, die für Optimierungsprobleme in Anwesenheit von zufälligen Einkünften entwickelt wurde, beweisen wir die Existenz und Eindeutigkeit der Lösung für das betrachtete Problem mit der logarithmischen Nutzenfunktion und einer Wahrscheinlichkeitsverteilung der Liquidationszeit unter minimalen Beschränkungen. Wir finden asymptotische Schranken für die Wertfunktion wenn die Liquidationszeit eine exponentielle oder eine Weibull - Verteilung hat. Wir finden optimale Strategien in der Feedbackform und zeigen wie sich diese von klassischen Mertonschen Strategien unterscheiden. Mit Hilfe von Lie Gruppenanalyse finden wir alle Lie Algebren für ein Problem mit einer allgemeinen Liquidationszeitverteilung in Fällen von HARA und logarithmischen Nutzenfunktionen und formulieren entsprechende Theoreme für alle diese Fälle.

Wir benutzen diese Lie Algebren um die untersuchten dreidimensionalen partiellen Differentialgleichungen auf Gleichungen niedrigerer Dimensionen zu reduzieren. Mehrere ähnliche Substitutionen wurden bereits früher in anderen Arbeiten benutzt, andere sind unserem Wissen nach neu. Der benutzte Methode der Lie Gruppenanalyse gibt uns die Möglichkeit, einen kompletten Satz von nicht äquivalenten Substitutionen und die entsprechenden reduzierten Gleichungen zu finden was für Probleme von diesem Typ bisher nicht möglich war. Es ist zu erwarten, dass weitere Untersuchungen dieser Gleichungen mit numerischen und qualitativen Methoden davon profitieren können.

To my grandfather Nikolay, my parents Olga and Pavel, and my wife Alexandra...

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Contents

List of Figures

List of Tables

1

Introduction

The global financial crisis of 2008-2009 revealed the significance of several problems that were well know on the global markets before but seemed unimportant and were regarded as the aspects that could have only a weak, if even noticeable, impact on the general situation. In particular, the financial crisis helped practitioners to understand the difficulties connected with a management of portfolios with illiquid assets and showed a significant need for solid mathematical models addressing this problem. Though financial institutes deal with illiquid assets on a regular basis there is no generally accepted framework for such portfolios especially if they provide stochastic income or down payments.

The most challenging task one faces defining such framework is to incorporate the illiquidity in a mathematically tractable way. Intuitively it is clear which of the assets we would call liquid, yet there is still no widely accepted way of defining illiquidity of an asset as a measurable parameter. In our previous work [48] we have tried to come up with a tractable empiric definition of illiquidity, but it is out of the scope of this work, since a mathematically correct definition, being a problem itself, is not the biggest challenge in this area. The exact formulation of the goals of the portfolio optimization is even more tedious, since illiquidity is usually connected with different sale mechanisms and with an essential liquidation time-lag. The stochastic processes that describe such effects are not studied profoundly in financial mathematics. Another important aspect is an overwhelming amount of empiric studies, that are constantly appearing, since data-sets, that describe illiquid assets, become as affordable as the computation power needed to work with them. One can find more and more works about certain type of illiquid assets and ad-hoc descriptions of their behavior, see, for example, [20] or [53]. Yet we see a lack of an integrated approach to the problems of that kind.

This thesis is devoted to one of the approaches to the problem of financial illiquidity, namely, an optimization of a portfolio with an illiquid asset sold in an exogenous random moment of time. We believe that the portfolio optimization framework could be a good unified method to work with the problems of illiquid assets. In this work we demonstrate how it can be advanced in an industrially applicable way.

The model that we formulate in this work draws attention to an interesting class of optimization problems that go in line with the so-called adapted resource allocation problem developed by *Pickenhain et al.* in [41]. The idea to regard an infinite horizon problem with certain weight function, see [51] and [52] for the details, seems very promising and fruitful. The problem of a portfolio optimization with an asset that has an exogenous random liquidation time that we regard in this work, could also be regarded as an infinite horizon problem with a special weight-function. This is a class of optimization problems that have direct connection with industrial needs and seems mathematically interesting in itself.

This work consists of eight chapters, including this, first one. In the second chapter we talk about the definition of illiquidity and make an overview of the existing approaches to the problem, discuss their advantages and disadvantages and give a deeper motivation for the necessity of a more mathematically precise and universal approach. In the third chapter we formulate the problem in the framework of portfolio optimization with an exogenous random liquidation time and regard the Hamilton-Jacobi-Bellmann equation that corresponds to our economic setting. We study the equation on the corresponding value function, prove the existence and uniqueness of the solution in a sense of viscosity functions for a broad class of liquidation time distributions of an illiquid asset. In the fourth chapter we discuss the difference of the optimal policies in the cases of different liquidation time distribution, namely, an exponential and Weibull distributed liquidation times in presence of a logarithmic utility function. In the fifth chapter we carry out the Lie group analysis of the HJB equations that arise in cases of different utility functions and different liquidation time distributions. We show that exponentially distributed liquidation time is a special case for the problems of this type, since it is the only liquidation time distribution for which a Lie algebra admitted by HJB equation is four dimensional. In Chapter 6 and Chapter 7 we list all possible reductions for the problem with HARA and logarithmic utility correspondingly and show that, indeed, for an exponential liquidation time the problem in both cases of HARA and logarithmic utility functions could be reduced to an ODE through a transformation of the admitted Lie group. In the last Chapter, Conclusion, we summarize all obtained results.

1. INTRODUCTION

2

Models of illiquidity

'Time – is money.'

Proverb

In this Chapter we make a brief overview of the approaches to the phenomenon of illiquidity. It is very important to note that there is an overwhelming amount of statistical analysis of the problem but the number of mathematical models that can aptly describe this issue is rather limited. Before we can talk about the models that could describe the phenomenon, we should try to define the phenomenon itself, since there is no academic definition that is considered as a standard one for financial mathematics let us briefly discuss the most popular ways to define and model illiquidity.

2.1 Ways to define liquidity

The understanding of the liquidity or, correspondingly, the illiquidity of a given asset is still a matter of a debate among practitioner as well as among academics. The intuitive idea that is standing behind is quite clear but the definition that could be not only mathematically correct and correspond to a given model but also could make sense from a practical point of view is a matter of a discussion.

Modern markets are constantly facing the problem of an optimal allocation (also sometimes referred as investment-consumption or allocation-consumption problem) for a portfolio that includes an illiquid asset, however, as far as we are concerned, there was no solid analytical solution for the problem of such type until 2006, when *Tebaldi and Schwartz* in [55] obtained such analytical solution using the framework of optimal portfolio allocation in presence of random income. The authors set up a finite investment horizon, denoted as T, and by design make an assumption that the investor sells the illiquid asset exactly in the end of this investment horizon T, moreover, the investor can not sell this asset in any moment of time, t, that precedes T. This models, being very interesting and novel, is unfortunately not that relevant for practical needs, since it is clearly applicable only in some exceptional cases (i.e. when the investor regards his human capital as an illiquid asset and knows exactly the day of his retirement) but generally we can not pre-determine the time when the opportunity to trade the illiquid asset occurs. This factor significantly reduces the industrial value of the mentioned research.

In order to form a better understanding of the problem of illiquidity for the reader, let us now retrospectively go through the history of this issue an list some of the most important and notable approaches that are popular among the practitioners and researchers in the field of finance.

The first definition of illiquidity, as far as we know, was given by *Keynes*, [36] as early as 1930: an asset is more liquid if it is 'more certainly realizable at short notice without loss'. For the next 50 years this compact definition was more or less the only one that was used in the scientific literature. This is only to be expected since is was widely assumed among the researchers that the problem was too complex on one hand and not that important industrially on the other. In the meantime the practitioners on the market were basically trusting their 'guts' and didn't go any deeper than an intuitive understanding of the issue.

Indeed, if we look, for example, into the 'Wall Street Words: An A to Z Guide to Investment Terms for Today's Investor' by *Scott* [56] we can find a representation of this practitioner's definition: *Illiquid asset is an asset that is difficult to buy or sell in a short period of time without its price being affected.* This definition is even shorter than the one proposed by Keynes, yet it is way more versatile. Though it gives us some practical understanding of the issue it can hardly be used in the mathematical applications because of its' qualitative character that definitely prevails over any quantitive approach here. Nevertheless, this definition points out several important aspects of the phenomenon that deserve our particular attention. First of all, it is the time aspect of the phenomenon: *difficult to buy or sell in a short period of time.* Since the definition does not qualitatively describe which period of time should be considered as a short one and which should be regarded as long, this aspect is defined through the design of the model that we to come up with. For example, as authors in [55] we could choose some constant time interval for which any time period smaller than this fixed predetermined constant could be a "short" one by definition. This approach is tempting since it is straight forward and rather easy to work with, however, it doesn't seem to be very fruitful since it does not capture a crucial aspect of the phenomenon: the buyer arrives stochastically for almost any asset that an individual wants to sell. The waiting time of a single deal can vary considerably under conditions that are very similar. The second important aspect of illiquidity is the actual price of an asset. It is clear that the buyer and the seller can have different estimation of the market conjuncture and therefore can have different opinions on a price that they would consider as a fair one for the same asset. This "misunderstanding" can result in the situation when the seller is ready to wait a bit more for a better price, which, in return increases the time horizon.

Time and price are two factors that play crucial roles when we talk about illiquidity, but there is a fundamental difference between the two. The time factor is defined by a general framework of the market model that we are using while the price depends, among others, on the investors' utility function.

In 1998 *Froot and Stein* [26] in a model that is considered classical for risk management practitioners up to now provide the following definition of the illiquid asset: *illiquid financial asset is an asset which, because of its information-intensive nature, cannot be frictionlessly traded in the capital markets.* Though this work was actively criticized in particular for its' "strictness", that could lead to some unexpected conclusions in certain circumstances (see, for example, [30]), it is frequently referred to in financial research that addresses illiquidity. For instance, in 2007 *Cao and Teïletche*, [15] were using this definition for the problem of the *alternative assets* (i.e. the assets that are different from core assets such as money market, bonds and equities, say, factories, immobilities, etc.). Between 2000 and 2005, the amount of such alternative assets under the management of hedge funds industry doubled and reached the $ 1,000 billions threshold. Naturally, alternative investments tend to be less liquid than the standard ones. The paper proposes insightful the idea of a management strategy for the portfolio that is comprised of these illiquid assets; however, the results are qualitative and the authors neither provides a full mathematical model nor give us an analytical solution of the problem.

The first attempt to describe the problem of illiquidity in a mathematically correct way and to provide a definition that would be in correspondence with the "empirical" understanding of this property was made in 1986 by *Lippman and McCall*, [40]. They defined the environment characterized by four different objects: c_i, T_i, X_i and β. All of them were described in the discrete time framework.

c_i is a cost of owning or operating the asset during the period number i. It can also be considered as the cost of the attempt to sell the asset. The offers come at every moment that is in the set $\{S_i : i = 1, 2, \dots\}$ of arrival times. These random variables S_i satisfy

$$S_i = \Sigma^i{}_{j=1} T_j,$$

where the integer valued random variables $T_i \geqslant 0$ neither need to be independent nor identically distributed.

X_i are positive independent identically distributed random variables that correspond to the price offered in the i-th moment. All the expenditures are discounted at the rate β so that a present value of a dollar received in period i is β^i. The discounted net receipts $R(\tau)$ associated with a stopping time τ is given by

$$\begin{aligned} R(\tau) &= \beta^\tau Y_{N(\tau)} - \Sigma^\tau{}_{j=1} \beta^i c_i, \qquad (2.1) \\ Y_i &= \begin{cases} X_i, & \text{if recall is not allowed,} \\ max(X_1, \dots, X_i), & \text{if recall is allowed,} \end{cases} \end{aligned}$$

where $N(\tau) = max\{n : S_n \leqslant \tau\}$ is the random number of offers that the seller observes when employing the decision rule τ and the random variable $Y_{N(\tau)}$ is the size of the accepted offer. Consequently, the seller chooses a stopping rule τ^* in the set T_i of all stopping rules such that

$$E[R(\tau^*)] = max\{E[R(\tau)] : \tau \in T_i\}.$$

Obviously, the time it takes to estimate the asset's value and to convert the asset into cash is defined by the random variable τ^*. *Lippman and McCall*, [40] proposed to regard the expectation of this variable, $E[\tau^*]$ as the measure of an assets' illiquidity. According to this definition as $E[\tau^*]$ increases (i.e. one need to wait longer until the asset is sold) a liquidity of a corresponding asset is to decrease.

This model was actually a milestone in this field and a number of the papers that try to describe illiquidity either use this approach directly or try to improve and broaden it. The serious disadvantage of this approach is that it does not take into consideration the other aspect of illiquidity that we have discussed above, that is this model hardly addresses the aspect of price and inevitable loss that is normally associated with the immediate need to sell an illiquid asset.

This aspect was mathematically described in 1994 by *Hooker and Kohn*, [31]. The authors decided to focus on the money aspect and to use the same framework (known as a search problem) but to make emphasis on the fact that an intention of a seller to sell an asset actually affects the price. The authors introduce an index of liquidity, so-called $\lambda(I_t)$, as

$$\lambda(I_t) = \frac{V(I_t) - L(I_t)}{V(I_t)},$$

where $V(I_t)$ is the value of the asset under optimal sale, as a function of the information set I_t and $L(I_t)$ is a loss from immediate sale of the asset. Since λ depends on the information set I_t they call this index the *conditional liquidity* of the asset. The authors also introduce the *expected liquidity* of an asset, Λ, which, naturally, is

$$\Lambda = E[\lambda(I_t)].$$

In their work *Hooker and Kohn*, [31] showed how their approach could be implemented on the capital market. They also gave an interesting intuitive example that could advocate their approach and show that it has more economic sense then the time-approach used before. Let us briefly tell about this reasoning since it gives a good understanding of the phenomenon.

Example 1. *Suppose the current price of an asset is $ 100, and the value of optimal sale is $ 100.01. The expected time to sale when following the optimal-sale policy is 100 years, so according to Lippman and McCall's definition the asset like that is highly illiquid. According to Hooker and Kohn it is almost perfectly liquid.*

In [48] we came up with another example for which *Lippman and McCall*'s approach would give a better description of the real situation on the market. Let us provide it here as well.

Example 2. *You have a factory which you want to sell. You have estimated the price of it and wait for a reasonable offer. Buyers when knowing that you want to have a deal get interested but do their own estimation. The offers they give come with a big time lag but could be really close to your estimated "optimal" price, therefore, according to Hooker and Kohn the asset could look rather liquid while our common sense tells us that it is not.*

This two simple examples let us grasp the biggest issue that concerns the problem of illiquidity. It seems that the offered price of the deal and the time lag between the start of the sale and the arrival of the offer should be both incorporated within a model. Moreover, in order to make this model applicable to the real problems neither the price not the time of the liquidation could be deterministic. That is why in this work we address an optimization problem for a portfolio that incorporates an illiquid asset that on one hand has a paper value, determined by a geometric Brownian motion, and on the other has an exogenous random liquidation time.

We have already mentioned above that there are a lot of other approaches to the illiquidity that are either based on the common sense or on the empirical facts and estimations or the combinations of the two. A majority of these approaches do a lot in common either with one or with the other definition provided above. For example, since the beginning of 2000s the idea to estimate the liquidity of the assets through the bid-ask spread became rather popular, see *Bangia et al. (1999)*, [5] or *Coppejans et al.* (2000), [18]. In fact this approach could be directly connected with the one proposed by *Hooker and Kohn*, [31]. Indeed, bid and ask are nothing more than the prices of the buyer and the seller correspondingly which means that their spread could be described via a ratio between the price by which we can sell the asset immediately and the price by which we desire to sell it.

The last research that is worth mentioning here is the paper of *E. Acar, R. Adams and R. Williams*, [2]. The authors propose to measure the illiquidity as "the ratio between volume and the distance moved by the market". They derive this concept from two basic ideas:

- The perfect liquidity indicator would assess the probable cost of the execution in the market.
- The perfect liquidity indicator would assess whether the markets were liable to anomalous moves.

They measure volume and price movement as evolving time series and then with a help of the empirical data they verify their approach. One disadvantage of this work is that the authors do not quantitatively describe any direct implications of such indicator on the investors' strategy. Another is the fact that many alternative assets are traded on the markets with little amount of market-data available, which makes a detection of certain 'anomalous' behavior a tedious task in itself.

Out of all the approaches described above there is one that deserves our special attention. It is a portfolio optimization problem for a portfolio with an illiquid asset on which we want to focus from now on.

2.2 State-of-the-art

In 1974 Miller in [43] formulated a problem of optimal consumption with a stochastic income stream. It was show that an upper bound on consumption is lower than the value of optimal consumption in the case where the random labor income is replaced by its mean. This was a first work to our knowledge that formulated the framework of a portfolio optimization in presence of a *stochastic income*.

In 1987 *Grossman and Baroque* in [28] analyzed a model of optimal consumption and portfolio selection in which consumption services are generated by holding a durable good. The durable good was considered illiquid in a sense that a transaction cost had to be paid when the good was sold. It was shown that it was optimal for the consumer to wait until a large change in wealth occurs before adjusting his consumption. As a consequence, the consumption based capital asset pricing model does not hold. At the same time it was demonstrated that the standard, one factor, market portfolio based capital asset pricing model does hold in this environment. This problem was very close to the problem of a *stochastic income* that we discuss later in detail, but this particular model was rather straight forward since it did not take into consideration any random effects associated with illiquidity such as stochastic liquidation time or price discount.

Two years later *Zeldes* provided the first numerical solution for the problem of optimal consumption with stochastic income and constant relative risk aversion in [61]. Among other results two quite important aspects were demonstrated. Consumption seemed to be highly sensitive to the transitory income and tended to grow in the case of a low risk-free interest.

Finally, in 1993 *Duffie and Zariphopoulou* in [21] develop the framework of the optimal consumption with undiversifiable income risk (also called a *stochastic income* model) as an extension for the continuous time model, proposed by *Merton*, [42]. They considered an infinite time horizon and proved the existence and uniqueness of the viscosity solution of the associated HJB equation for the class of concave utility functions $U(c)$ satisfying the following conditions: U in c is strictly concave; $C^2(0,+\infty)$, $U(c) \leq M(1+c)^\gamma$, with $0<\gamma<1, M>0$; $U(0)\geq 0$, $\lim\limits_{c\to 0} U'(c) = +\infty$, $\lim\limits_{c\to\infty} U'(c) = 0$.

Later, in 1997, in [22] an extended problem of hedging in incomplete markets with hyperbolic absolute risk aversion (so called HARA) utility function was studied. Here the stochastic income cannot be replicated by trading available securities. An investor receives stochastic income in moment t at a rate Y_t, where

$$dY_t = \mu Y_t dt + \eta Y_t dW_t^1, \quad t \geq 0, Y_0 = y, \quad y \geq 0$$

and $\mu, \eta > 0 - const$ here W^1 is a standard Brownian motion. The riskless bank account has a constant continuously compound interest rate r. A traded security has a price S given by

$$dS_t = \alpha S_t dt + \sigma S_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2),$$

$\alpha,\ \sigma > 0 - const$ and W^2 is an independent standard Brownian motion, $\rho \in (-1,1)$ is a correlation between price processes S_t and Y_t. The investor utility function for consumption process c_t is given by

$$\mathcal{U}(c(t)) = E\left[\int_0^\infty e^{-\kappa t} U(c(t))dt\right], \quad U(c(t)) = c(t)^\gamma,$$

where $\gamma \in (0,1)$ and κ is a discount factor $\kappa > r$.

Remark 1. *The notation of the strategy (π, c) is standard for the problems of such kind. Throughout this work we will denote the amount of the investment in a liquid risky asset as π and investor's consumption as c. Both controls do depend on time, so to emphasize it to the reader we might also use $(\pi(t), c(t))$ or even (π_t, c_t) from time to time.*

The investors wealth process L evolves

$$dL_t = [rL_t + (\alpha + \delta - r)\pi_t - c_t + Y_t]dt + \sigma\pi_t(\rho dW_t^1 + \sqrt{1-\rho^2}\, dW_t^2), \quad t \geq 0, L_0 = l,$$

where δ could be regarded as the dividends payed constantly from an illiquid asset or as the possession costs, l is an initial wealth endowment and π_t represents an investment in the risky asset S, with the remaining wealth held in riskless borrowing or lending. The goal is to characterize an investor value function $V(l, y) = \sup_{(\pi,c)\in\mathcal{A}(l,y)} \mathcal{U}(c)$. The set $\mathcal{A}(l, y)$ is a set of admissible controls (π, c) such that $L_t \geq 0$.

The authors in [22] proved the smoothness of the viscosity solution of the associated Hamilton-Jacobi-Bellman (HJB) equation in the case of the HARA utility function and the infinite time horizon. This proof heavily relies on a reduction of the initial HJB equation to an ODE. After this reduction the main result follows from the uniform convergence of the classical solution of a uniformly elliptic equation to the viscosity solution, which is unique.

It is important to mention that the authors use the discount factor $e^{-\kappa t}$ in [22] as a technical factor which is not related to stochastic income. The economical setting does not imply any liquidation of an illiquid asset which provides stochastic income Y_t.

In 2007 *Schwarz and Tebaldi* in [55] broadened a model of random income proposed before and connected it to the problems of illiquidity. They assumed that the non-traded illiquid asset generates a flow of random income in the form of dividends, until it is sold at a fixed moment of time. This idea allowed to build models for a portfolios with illiquid assets, using the results obtained for the problems with random income. One of a huge challenges connected with optimizations problems in presence of illiquidity is the question of pricing of illiquid assets that is a serious mathematical problem in itself. Assuming that the asset generates a certain dividends, connected with its fair price authors could elegantly incorporate illiquid asset in the model. Further, the authors define illiquid asset as an asset that can not be sold neither piece by piece nor at once before the investment's horizon, denoted as T, which is a fixed deterministic value at which the asset generates a random cash-flow equal to its' paper-value at this moment T (the cash-flow is denoted as H_T). With this economical reasoning behind it this model of illiquidity looks rather promising yet needs a more exact qualitative and quantitive description. In [48] we have broadened this framework for the case of logarithmic utility and finite deterministic liquidation time. In this particular work we talk about a further improvement of this framework, especially, weakening the trading conditions for an illiquid asset that can move a model closer to the practical needs.

Later in 2008 *Schwarz et al.* in [16] applied the approach very close to the one formulated in [55] to the problem of housing choice for a household. The constrained of a deterministic time was abandoned as the idea of the model was to compare two 'realities': one, where housing was purely illiquid and another 'thought-experiment' reality, where the household could sell the real-estate partially. It was demonstrated that optimal strategies for two models differ significantly.

One of the possible extensions of this problem was done by *Ang, Papanikolaou and Westerfeld* in [4]. They considered exactly the same model as in [55]. However, they assumed that an illiquid asset can be traded but only at infrequent, stochastic moments of time and thus the whole three-asset portfolio could be rebalanced. With a series of numerical calculations they provide an intuition of the influence of illiquidity on the marginal utility of the investor. The authors numerically study the cases when amount of the illiquid wealth is significantly bigger than the amount of the liquid capital and comparing it with the opposite case (insignificantly small amount of illiquid wealth) they show that the effects of the asset being illiquid may cause unbounded deviations from the Merton solution.

In 2008 *He* [29] proposed a model with the same set-up but different constraints on illiquid asset. While the investor can instantaneously transfer funds from the liquid to the illiquid asset, the vice versa transaction is allowed only in exponentially distributed moments of time. The author finds an approximate numerical solution of the problem for the constant risk-aversion (CARA) utility function.

Industry is highly interested in feasible illiquidity models. The practitioners constantly state that portfolios that include illiquid assets have a heavily time-dependent behavior (see, for example, [15]). There were several attempts made in this direction. In [34], for example, the authors use endogenous random time horizon and demonstrate that a standard optimization problem with an endogenous stopping time differs from classical Metron case.

In this work we focus on the time-horizon is an exogenous random variable. We would like to note that the set-up with exogenous time is actually economically motivated. For example, standard inheritance procedures in several EU countries assume that the illiquid assets are sold and the cash is then divided between the heirs. Naturally the sale occurs in a random moment of time and the inheritance manager splits the cash between the heirs immediately after the sale. Another example of an exogenous liquidation time that justifies our model are shares-for-the-loan auctions. This phenomenon is typical for the the emerging markets where governmentally owned businesses are at some point privatized fully or partially. For example, it was very typical for a post-soviet markets in their transition period and is still relevant for a number of states in the Eastern Europe, [58].

3

Portfolio optimization in the case of an illiquid asset with a given liquidation time distribution. General case.

3.1 Economical setting

We assume that the investor's portfolio includes a riskless bond, a risky asset and a non-traded asset that generates stochastic income, i.e. dividends. However, in contrast with the previous works we replace the liquidation time that was deterministic before with a stochastically distributed time T. A risk-free bank account B_t with the interest rate r and a stock price S_t describe classical Black-Scholes market [8]

$$dB_t = rB_t dt, \quad dS_t = S_t(\alpha dt + \sigma dW_t^1),\ t \leq T \tag{3.1}$$

where the interest rate r, the continuously compounded rate of return $\alpha > r$ and the standard deviation σ are assumed to be constant; $r, \alpha, \sigma - const.$ An illiquid asset H_t that can not be traded up to the time T and which paper value is correlated with the stock price and follows

$$\frac{dH_t}{H_t} = (\mu - \delta)dt + \eta(\rho dW^1 + \sqrt{1-\rho^2}dW^2),\ t \leq T. \tag{3.2}$$

where μ is the expected rate of return of the risky illiquid asset, (W^1, W^2) are two independent standard Brownian motions, δ is the rate of dividend paid by the illiquid asset, η is the continuous standard deviation of the rate of return, and $\rho \in (-1; 1)$ is the correlation coefficient between the stock index and the illiquid risky asset. The parameters μ, δ, η, ρ are all assumed to be constant. The liquidation time T is an exogenous random-distributed continuous variable which does not depend on the Brownian motions (W^1, W^2). The probability density function of liquidation time distribution T is denoted by $\phi(t)$ whereas $\Phi(t)$ denotes the cumulative distribution function, and $\overline{\Phi}(t)$ the survival function also known as *a reliability function* $\overline{\Phi}(t) = 1 - \Phi(t)$. We omit here the explicit notion of the possible parameters of distribution in order to make the formulae shorter.

Given the filtration $\{\mathcal{F}_t\}$ generated by the Brownian motion $W = (W^1, W^2)$ we assume that the consumption process is an element of the space $\mathcal{L}_+$ of non-negative $\{\mathcal{F}_t\}$-progressively measurable processes c_t such that

$$E\left(\int_0^s c(t)dt\right) < \infty, \quad s \in [0, T], \tag{3.3}$$

where E denotes a mathematical expectation with respect to filtration $\{\mathcal{F}_t\}$. The investor wants to maximize the average utility consumed up to the time of liquidation, given by

$$\mathcal{U}(c) := \mathcal{E}\left[\int_0^T U(c(t))dt\right]. \tag{3.4}$$

Here we used $\mathcal{E}$ to indicate that we are averaging over all random variables including T. The wealth process L_t is the sum of cash holdings in bonds, stocks and *random* dividends from the non-traded asset minus the consumption stream. Thus, we can write

$$dL_t \;=\; (rL_t + \delta H_t + \pi_t(\alpha - r) - c_t)dt + \pi_t \sigma dW_t^1. \tag{3.5}$$

The set of admissible policies is standard and consists of investment strategies (π_t, c_t) such that

1. c_t belongs to $\mathcal{L}_+$,

2. π_t is $\{\mathcal{F}_t\}$-progressively measurable and $\int_t^s (\pi_\tau)^2 d\tau < \infty$ a.s. for any $t \le \tau \le T$,

3. L_t, defined by the stochastic differential equation (3.5) and initial conditions $L_t = l > 0$, $H_t = h > 0$ a.e. $(t \leq T)$.

We claim that one can explicitly average (3.4) over T and with the certain conditions posed on $\overline{\Phi}$ and $U(c)$ the problem (3.4) is equivalent to the maximization of

$$\mathcal{U}(c) := E\left[\int_0^\infty \overline{\Phi}(t)U(c(t))dt\right], \tag{3.6}$$

where E is an expectation over space coordinates excluding T.

Remark 2. *It is important to note, that if T is exponentially distributed we get* precisely *the problem of optimal consumption with random income that was studied in [22] and already discussed in the introduction.*

Remark 3. *The idea to work with a non-exponential discounting is not new, for example Ivar Ekeland in [23], [24] has shown the possibility to work with different discounting factors. Skiba and Tobacman also mention a non-exponential discounting in [57] in the loan context, yet the authors do not provide any mathematically strict way to model these effects. To our knowledge the idea of a discounting different from an exponential one in a framework of illiquidity was never proposed before.*

We demonstrate here a formal derivation of the equivalence between two optimal problems briefly mentioned by Merton in [42].

Proposition 1. *The problems (3.4) and (3.6) are equivalent provided*

$$\lim_{t\to\infty} \overline{\Phi}(t)E\left[U(c(t)\right] = 0. \tag{3.7}$$

Proof. We have

$$\begin{aligned} \mathcal{E}\left[\int_0^T U(c(t))dt\right] &= \int_0^\infty \phi(T)E\left[\int_0^T U(c(t))dt\right]dT \\ &= \int_0^\infty \int_0^T \phi(T)g(t)dTdt, \end{aligned} \tag{3.8}$$

where $g(t) = E[U(c(t)]$. Because of the absolute convergence $\mathcal{E}\left[\int_0^T U(c(t))dt\right] =$

$\int_0^T g(t)dt$ and integrating (3.8) by parts we get

$$\begin{aligned}\int_0^\infty \int_0^T \phi(T)g(t)dTdt &= \overline{\Phi}(T)\int_0^T g(t)dt\Big|_0^\infty + \int_0^\infty \overline{\Phi}(t)g(t)dt \\ = E\left[\int_0^\infty \overline{\Phi}(t)U(c(t))dt\right], & \qquad (3.9)\end{aligned}$$

where we used the condition (3.7) to eliminate the first term, and the absolute convergence of the integral to move the expectation out. • □

Remark 4. *In the majority of the models consumption $c(t)$ is bounded as time goes to infinity. For all these models condition (3.7) is satisfied automatically. Yet if one regards absolute values of consumption and it grows as time goes to infinity this constraint is needed.*

From now on in this work we will regard the problem (3.4) with random liquidation time T that has a distribution satisfying the condition (3.7) in Proposition 1 and, therefore, corresponds to the *value function* $V(t,l,h)$ which is defined as

$$V(t,l,h) = \max_{(\pi,c)} E\left[\int_t^\infty \overline{\Phi}(\tau)U(c(\tau))d\tau | L(t)=l, H(t)=h\right]. \qquad (3.10)$$

For the value function we can derive a HJB equation on which we focus in this work

$$\begin{aligned}V_t(t,l,h) &+ \frac{1}{2}\eta^2h^2V_{hh}(t,l,h) + (rl+h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ &+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0, \qquad (3.11)\\ G[\pi] &= \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h \\ &+ \pi(\alpha-r)V_l(t,l,h), \qquad (3.12)\\ H[c] &= -cV_l(t,l,h) + \overline{\Phi}(t)U(c), \qquad (3.13)\end{aligned}$$

with the boundary condition

$$V(t,l,h) \to 0, \text{ as } t \to \infty.$$

The fact that the value function for a problem of such kind can be regarded as a viscosity solution is well known (see e.g. [19]). Before we move further we need to introduce some basic ideas and principles concerning viscosity solutions.

We address an interested reader to the book of *Fleming and Soner* [25] for the detailed and fundamental coverage of this topic. For the ones interested in the connection between viscosity solutions and optimization problems we also can recommend [6]. In this work we will use the notations form [19]

3.2 Introduction to viscosity solutions

Generally speaking, viscosity solution techniques can be applied to PDEs having the form $F(x,u,Du,D^2u)=0$ where $F:\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\times\mathcal{S}(N)\to\mathbb{R}$ and $\mathcal{S}(N)$ is the set of symmetric matrices $N\times N$. This approach has several crucial advantages

- continuous functions can be solutions of fully nonlinear equations of second order,
- very general existence and uniqueness theorems can be formulated and have relatively simple proofs,
- general boundary conditions can be obtained.

What is especially important is that this approach has a great flexibility in passing to limits in various settings. In $F(x,u,Du,D^2u)$ function u is a real-valued function defined on some subset $\mathcal{O}$ of $\mathbb{R}^N$, generally, Du corresponds to the gradient of u and D^2u corresponds to the matrix of second derivatives of u, but Du and D^2u do not have to have classical meanings.

In order to apply the theory to a given equation $F=0$, one should require that F satisfies a fundamental monotonicity condition formulated in [19] as follows

$$F(x,r,p,X)\le F(x,s,p,Y),\quad \textit{whenever}\;\; r\le s\;\; \textit{and}\;\; Y\le X; \tag{3.14}$$

where $r,s\in\mathbb{R},\quad x,p\in\mathbb{R}^N,\quad X,Y\in\mathcal{S}(N)$ and $\mathcal{S}(N)$ is equipped with its usual order.

Splitting condition (3.14) in two we obtain

$$\begin{aligned} F(x,r,p,X) &\le F(x,s,p,X)\;\; \textit{when}\;\; r\le s, && (3.15)\\ F(x,r,p,X) &\le F(x,r,p,Y)\;\; \textit{when}\;\; Y\le X. && (3.16)\end{aligned}$$

Condition (3.16) is usually called *degenerate ellipticity*. The function F is called degenerate elliptic if (3.16) holds. If both conditions (3.15) and (3.16) hold F is called *proper*.

In order to apply viscosity solution approach one needs to assume that F is proper, i.e. satisfies (3.14) and continuous, if it is not stated otherwise. In order to show a motivation that stands behind these notions let us, for a start, assume that u is C^2 (i.e., twice continuously differentiable) on $\mathbb{R}^N$ and

$$F(x, u(x), Du(x), D^2u(x)) \leq 0$$

holds for any x. This means that u is a subsolution of $F = 0$ in a classical sense or, equivalently, a classical solution of $F \leq 0$ in $\mathbb{R}^N$. Let us also assume that ζ is C^2 and there is a local maximum of $u - \zeta$ that is reached in $\hat{x}$. We can immediately see that $Du(\hat{x}) = D\zeta(\hat{x})$ and $D^2u(\hat{x}) \leq D^2\zeta(\hat{x})$ and using (3.16) we obtain

$$F(x, u(\hat{x}), D\zeta(\hat{x}), D^2\zeta(\hat{x})) \leq F(x, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x})) \leq 0. \tag{3.17}$$

Let us note that extremes of this inequality do not depend on the derivatives of u. This allows us to define an arbitrary function u to be a generalized subsolution of $F = 0$ in certain sense if

$$F(x, u(\hat{x}), D\zeta(\hat{x}), D^2\zeta(\hat{x})) \leq 0, \tag{3.18}$$

keeping in mind that ζ is C^2 and $\hat{x}$ is a local maximum of $u - \zeta$.

Before moving on to a formal definition, let us point out that x is near $\hat{x}$ and ζ is C^2 we see that $u(x) \leq u(\hat{x}) - \zeta(\hat{x}) + \zeta(x)$. Using Taylor approximation we obtain

$$u(x) \leq u(\hat{x}) + \langle p, x - \hat{x}\rangle + \frac{1}{2}\langle X(x - \hat{x}), x - \hat{x}\rangle + o(|x - \hat{x}|^2) \quad as \quad x \to \hat{x} \tag{3.19}$$

where $p = D\zeta(\hat{x})$ and $X = D^2\zeta(\hat{x})$ and $\langle \cdot, \cdot \rangle$ denotes scalar product. If (3.18) holds for some $(p, X) \in \mathbb{R}^N \times \mathcal{S}(N)$ and u is C^2 at $\hat{x}$, then $p = Du(\hat{x})$ and $D^2u(\hat{x}) \leq X$. This basically means that u is a solution of $F \leq 0$ in a classical sense, then $F(\hat{x}, u(\hat{x}), p, X) \leq 0$ holds wherever (3.18) is satisfied. A definition for non-differentiable solutions of inequality $F \leq 0$ could be based on this fact. Technically, working with (3.18) would lead us to notions obtained in a context of function ζ instead of notions for the non-differentiable function u, whereas working with (3.19) could give us a clue on how could we define 'Du, D^2u' in the case of non-differentiable u. So, let us focus on (3.19) and develop the reasoning in the following way.

Let us introduce a set $\mathcal{O} \subset \mathbb{R}^N$ on which $F \leq 0$ and we can regard inequalities like (3.19) on this $\mathcal{O}$. For a start this set $\mathcal{O}$ can be arbitrary but further we require it to be locally compact. Now if $u : \mathcal{O} \to \mathbb{R}$, $\hat{x} \in \mathcal{O}$ and (3.19) is satisfied as $\mathcal{O} \ni x \to \hat{x}$ we say $(p, X) \in J^{2,+}_{\mathcal{O}} u(\hat{x})$ (i.e. the second-order 'superjet' of u at $\hat{x}$). This way we define a mapping $J^{2,+}_{\mathcal{O}} u$ from $\mathcal{O}$ to the subsets of $\mathbb{R}^N \times \mathcal{S}(N)$.

Let us here give an example of such $J^{2,+}_{\mathcal{O}} u$ provided in [19] verbatim.

Example 3. *[19] If u is defined on $\mathbb{R}$ by*

$$u(x) \quad = \quad \begin{cases} 0, & \text{for } x \leq 0, \\ ax + \frac{b}{2}x^2 & \text{for } x \geqslant 0, \end{cases}$$

then $J^{2,+}_{[-1,0]} u(0) = ((-\infty, 0) \times \mathbb{R}) \cup (\{0\} \times [0, \infty))$, *while*

$$J^{2,+}_{\mathbb{R}} u(0) \quad = \quad \begin{cases} \emptyset, & \text{if } a > 0, \\ \{0\} \times [\max\{0, b\}, \infty) & \text{if } a = 0, \\ ((a, 0) \times \mathbb{R}) \cup (\{0\} \times [0, \infty)) \cup (\{a\} \times [b, \infty)) & \text{if } a < 0, \end{cases}$$

This example illustrates that $J^{2,+}_{\mathcal{O}} u(x)$ does indeed depend on $\mathcal{O}$ but is, in fact, the same for all sets $\mathcal{O}$ for which x is an interior point. Let us denote this common value as $J^{2,+}u(x)$. Now we can repeat the same reasoning but for another sign of inequality (3.19), i.e.

$$u(x) \geqslant u(\hat{x}) + \langle p, x - \hat{x} \rangle + \frac{1}{2} \langle X(x - \hat{x}), x - \hat{x} \rangle + o(|x - \hat{x}|^2) \quad as \quad x \to \hat{x}, \tag{3.20}$$

and in the same manner define the second-order 'subjets' $J^{2,-}_{\mathcal{O}} u$ and, correspondingly $J^{2,-}u$. Let us note that $J^{2,-}_{\mathcal{O}} u(x) = -J^{2,-}_{\mathcal{O}}(-u)(x)$.

It is also important to introduce the notations from [19]

$$USC(\mathcal{O}) = \{\text{upper semicontinuous functions } u : \mathcal{O} \to \mathbb{R} \ \},$$
$$LSC(\mathcal{O}) = \{\text{lower semicontinuous functions } u : \mathcal{O} \to \mathbb{R} \ \}.$$

Now, when we covered these general principles and notions, we can introduce the concepts of viscosity subsolutions, supersolutions, and solutions as they are defined in [19].

Definition 1. *Let F satisfy (3.14) and $\mathcal{O} \subset \mathbb{R}^N$. A* viscosity subsolution *of $F = 0$ or, equivalently, a viscosity solution of $F \leq 0$ on $\mathcal{O}$ is a function $u \in USC(\mathcal{O})$ such that*

$$F(x, u(x), p, X) \leq 0$$

for all $x \in \mathcal{O}$ and $(p, X) \in J^{2,+}_{\mathcal{O}} u(x)$.

Similarly, a viscosity supersolution *of $F = 0$ on $\mathcal{O}$ is a function $u \in LSC(\mathcal{O})$ such that*

$$F(x, u(x), p, X) \geqslant 0$$

for all $x \in \mathcal{O}$ and $(p, X) \in J^{2,-}_{\mathcal{O}} u(x)$.

Finally, u is a viscosity solution *of $F = 0$ in $\mathcal{O}$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $\mathcal{O}$.*

3.3 Viscosity solution of the optimization problem for a portfolio with illiquid asset. Comparison Principle

Generally, the value function for a problem of such kind as (3.11) is a viscosity solution if the control and state variables are uniformly bounded. However, this is not the case for the optimal consumption problem and thus a more sophisticated proof is needed. This problem was previously studied in [21], [22] and [60]. The main difficulties in our case come from the non-exponential time discounting we are using in the utility functional (3.10). As we mentioned before, this leads to the HJB equation (3.11) being three dimensional. This demands additional work. We will concentrate on the new results and will omit the details of the arguments that work in our problem and could be found in [21].

Theorem 1. *There exists a unique viscosity solution of the corresponding HJB equation (3.10) if*

1. *$U(c)$ is strictly increasing, concave and twice differentiable in c,*

2. $\lim_{t\to\infty} \overline{\Phi}(t)E[U(c(t))] = 0$, $\overline{\Phi}(t) \sim e^{-\kappa t}$ *or faster as* $t \to \infty$,

3. $U(c) \leq M(1+c)^{\gamma}$ *with* $0 < \gamma < 1$ *and* $M > 0$,

4. $\lim_{c\to 0} U'(c) = +\infty$, $lim_{c\to+\infty} U'(c) = 0$.

The proof of this statement is to be done in three steps. At first we need to establish certain properties of the value function $V(t, l, h)$ that corresponds to our problem. These properties are formulated and proved in Lemma 1 that follows. Then we show that the value function with such properties is a viscosity solution of the problem, this is done in Lemma 2. The uniqueness of this solution follows from the *comparison principle* that is actually a very useful tool by itself and is formulated and proved in Theorem 2.

Lemma 1. *Under the conditions* (1) – (4) *from Theorem 1 the value function* $V(t, l, h)$ *(3.10) has the following properties:*

1. $V(t, l, h)$ *is concave and non-decreasing in* l *and in* h,

2. $V(t, l, h)$ *is strictly increasing in* l,

3. $V(t, l, h)$ *is strictly decreasing in* t *starting from some point,*

4. $0 \leq V(t, l, h) \leq O(|l|^{\gamma} + |h|^{\gamma})$ *uniformly in* t.

1. *Proof.* Let us look on the points (l_1, h_1) and (l_2, h_2) with corresponding $(\pi_1^{\epsilon}, c_1^{\epsilon})$ and $(\pi_2^{\epsilon}, c_2^{\epsilon})$ which are ϵ-optimal controls in each of this points respectively or in another words:

$$V(t, l, h) \leq E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^{\epsilon})d\tau\right] + \epsilon,$$

where $l = l_1, l_2$, $h = h_1, h_2$ and $c = c_1, c_2$ correspondingly. We choose the point $(\alpha c_1^{\epsilon} + (1-\alpha)c_2^{\epsilon})$, where $\alpha \in \mathbb{R}$ and $0 < \alpha < 1$. The policy $(\alpha l_1 + (1-\alpha)l_2, \alpha h_1 + (1-\alpha)h_2)$ is admissible for this point

$$\begin{aligned} & V(t, \alpha l_1 + (1-\alpha)l_2, \alpha h_1 + (1-\alpha)h_2) \\ \geqslant \; & E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(\alpha c_1^{\epsilon} + (1-\alpha)c_2^{\epsilon})d\tau\right]. \end{aligned} \tag{3.21}$$

The utility function is concave (see condition 1. from Theorem 1), so we can write

$$E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(\alpha c_1^\epsilon+(1-\alpha)c_2^\epsilon)d\tau\right]\geqslant \alpha E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(c_1^\epsilon)d\tau\right]$$
$$+\quad (1-\alpha)E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(c_2^\epsilon)d\tau\right]\geqslant \alpha V(t,l_1,h_1)+(1-\alpha)V(t,l_2,h_2)+2\epsilon.$$

Now that we have proved the concavity of $V(t,l,h)$ in l and h. We can show that it is not decreasing. Without any loss of generality we can assume that $l_1\leqslant l_2$ and $h_1\leqslant h_2$. Note that if $(\pi_1^\epsilon,c_1^\epsilon)$ is ϵ-optimal for (l_1,h_1) it is admissible for (l_2,h_2) which means that

$$V(t,l_1,h_1)\leqslant V(t,l_2,h_2)+\epsilon,$$

setting $\epsilon\to 0$ we get that $V(t,l,h)$ is non-decreasing in first two variables.

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2. *Proof.* To show that $V(t,l,h)$ is strictly increasing in l we can assume the contrary. Let us look at $l_1<l_2$ such that $V(t,l_1,h)=V(t,l_2,h)$. Since we already know that $V(t,l,h)$ is non-decreasing in l the function V should be constant on the interval $[l_1,l_2]$, moreover, since V is concave in l this interval has to be infinite. This means that there is such l_0 that $V(t,l,h)=V(t,l_0,h)$ for any $l\geqslant l_0$. Let $(\pi^\epsilon,c^\epsilon)$ be ϵ-optimal for (t,l_0,h)

$$V(t,l_0,h)\le E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(c^\epsilon)d\tau\right]+\epsilon. \tag{3.22}$$

We denote $\int_t^{+\infty}\overline{\Phi}(\tau)d\tau$ as $K(t)$ and look on the inequality

$$l_1>\max\left(l_0,U^{-1}\left(\frac{E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(c^\epsilon)d\tau\right]+\epsilon}{K(t)}\right)/r\right),$$

where U^{-1} denotes an inverse utility function. The strategy $\pi=0$ and $c=rl_1$ does not depend on time but is admissible for (t,l_1,h). Indeed, due to the fact that the strategy $(0,rl_1)$ does not depend on time one can write

$$K(t)U(rl_1)=E\left[\int_t^{+\infty}\overline{\Phi}(\tau)U(rl_1)d\tau\right]\leqslant V(t,l_1,h).$$

But if we look at $K(t)U(rl_1)$ and use the formula for l_1 given above we get

$$K(t)U(rl_1) > E\left[\int_t^{+\infty} \overline{\Phi}(\tau)U(c^\epsilon)d\tau\right] + \epsilon,$$

which is greater or equal to $V(t, l_0, h)$ according to the Equation (3.22). That gives us $V(t, l_0, h) < V(t, l_1, h)$ which is a contradiction keeping in mind that $l_1 > l_0$. So, V is strictly increasing in l. • □

3. *Proof.* According to condition 2 from Theorem 1 the product of $\overline{\Phi}(t)$ and $U(c(t))$ as well as $\overline{\Phi}(t)$ itself should be both decreasing for $t > \tau$ starting from a large enough τ. So we choose two moments of time t_1 and t_2 such that $\tau < t_1 < t_2$, $\Delta t = t_2 - t_1$ and look at $V(t_2, l, h)$ then

$$V(t_2, l, h) = \int_{t_2}^{\infty} \overline{\Phi}(t)U(c_t)dt \overset{\tau=t-\Delta t}{=} \int_{t_1}^{\infty} \overline{\Phi}(\tau + \Delta t)U(c_{\tau+\Delta t})d\tau,$$

since $\overline{\Phi}(t)$ is decreasing for every $t > t_1$ and the process $c_{\tau+\Delta t}$ for $\tau \geq t_1$ with $L(t_2) = l, H(t_2) = h$ has exactly the same realizations as c_τ for $\tau \geq t_1$ with $L(t_1) = l, H(t_1) = h$ one can write

$$\int_{t_1}^{\infty} \overline{\Phi}(\tau + \Delta t)U(c_{\tau+\Delta t})d\tau < \int_{t_1}^{\infty} \overline{\Phi}(\tau)U(c_{\tau+\Delta t})d\tau \leqslant V(t_1, l, h).$$

So for any t_1 and t_2 such that $\tau < t_1 < t_2$ we get $V(t_1, l, h) > V(t_2, l, h)$.
• □

4. *Proof.* Instead of the original problem with the non-traded income generated by $H_t, H_0 = h$ one can consider a fiction consumption-investment problem with a special asset on the market, such that has a sufficient initial endowment (meaning that one can generate exactly the same income flow as H_t would by investing in the market). Suppose the synthetic asset follows geometrical Brownian motion

$$dS'_t = \alpha' S'_t + \sigma' S'_t dW_t, \quad t \geq 0 \;\; S'_0 = s', \quad s' > 0, \tag{3.23}$$

with constants α' and σ' to be defined later. Next, the *initial wealth equivalent* of the stochastic income is defined by

$$\begin{aligned} f(h) &= \delta E_h\left[\int_0^\infty e^{-\kappa t}\xi_t H_t dt\right], \\ \xi_t &= \exp\left(-\frac{1}{2}(\theta_1^2 + \theta_2^2) + \theta_1 W_t^1 + \theta_2 W_t^2\right), \end{aligned}$$

where $\theta_1 = (\alpha - r)/\sigma$ and $\theta_2 = (\alpha' - r)/\sigma'$.

It turns out that with the properly chosen α' and σ' we achieve that $f(h) < C_1 h$. Moreover, the stochastic income rate H_t can be replicated by a self-financing strategy on the complete market (B_t, S_t, S'_t) with the additional initial endowment $f(h)$. This fact is well known from the martingale-based studies of the consumption-investment problem, primarily carried out in [32] and [35].

To finish the proof, we notice that since the stochastic income can be replicated, any admissible strategy for the original problem with initial conditions (l, h) is dominated by a strategy on the synthetic market with initial endowment $l + f(h) < l + C_1 h$. On the other hand, we have the growth conditions for $\overline{\Phi}(t)$ and $U(c)$. So, the maximal utility is bounded from above by the solution of the classic investment-consumption problem with initial wealth $l + C_1 h$, HARA utility and exponential discounting. Due to Merton we have a closed form solution for this case. Putting everything together, we obtain the desired bound (all the further details can be found in [32] and [21]). • □

Now we can prove the existence of the viscosity solution of the problem (3.11).

Lemma 2. *Under the conditions of Lemma 1 the function $V(t, l, h)$ is a viscosity solution of (3.11) on the domain $D = (0, \infty) \times (0, \infty) \times (0, \infty)$.*

Proof. We again use the reasoning from the proof of Theorem 4.1 in [21] but modify it for our case. To show that V is a viscosity solution one need to show that it is a viscosity supersolution and a viscosity subsolution of the problem.
Let us show at first that $V(t, l, h)$ is a viscosity supersolution for (3.11). Let us look at $\phi \in C^2(D)$ and assume that $(t_0, l_0, h_0) \in D$ is a point where a minimum of $V - \phi$ is achieved. We can assume that $V(t_0, l_0, h_0) = \phi(t_0, l_0, h_0)$ and $V > \phi$ in D without any loss of generality. To show that V is a supersolution we need to check that $\mathcal{J}[\phi](t_0, l_0, h_0, \pi, c) \leq 0$, where

$$\begin{aligned}\mathcal{J}[\phi](t_0, l_0, h_0, \pi, c) = {} & \phi_t(t_0, l_0, h_0) + \frac{1}{2}\eta^2 h_0^2 \phi_{hh}(t_0, l_0, h_0) + (r l_0 + \delta h_0)\phi_l(t_0, l_0, h_0) \\ & +(\mu - \delta) h_0 \phi_h(t_0, l_0, h_0) + \max_\pi G[t_0, l_0, h_0, \pi] + \max_c H[t_0, l_0, h_0, c],\end{aligned}$$

with $G[\pi]$ and $H[c]$ defined in (3.11).

We consider a locally constant strategy (π_0, c_0) for the period of time θ tending to zero. One can take $\theta = \min\{1/n, \tau\}$ where $\tau = \inf\{t \geq t_0 : W_t = 0\}$ to ensure feasibility of this strategy. Since this strategy is suboptimal we can write (using the dynamic programming principle, [25])

$$\begin{aligned} V(t_0, l_0, h_0) &\geq E\left[\int_{t_0}^{t_0+\theta} \overline{\Phi}(t)U(c_0)dt + V(L_\theta, H_\theta, \theta)\right] \\ &\geq E\left[\int_{t_0}^{t_0+\theta} \overline{\Phi}(t)U(c_0)dt + \phi(L_\theta, H_\theta, \theta)\right]. \end{aligned} \tag{3.24}$$

On the other hand, applying Itô calculus to the smooth function ϕ we can expand

$$E[\phi(\theta, L_\theta, H_\theta)] = \phi(t_0, l_0, h_0) + E\left[\int_{t_0}^{t_0+\theta} D\phi(s, L_s, H_s)ds\right].$$

Substituting into (3.24) and using standard estimates to approximate the terms with $\phi(s, l_s, h_s)$, $\phi_l(s, l_s, h_s)$, $\phi_h(s, l_s, h_s)$, etc. via $\phi(t_0, l_0, h_0) + O(s)$, $\phi_l(t_0, l_0, h_0) + O(s)$, $\phi_h(t_0, l_0, h_0) + O(s)$ respectively, we obtain the bound

$$E\left[\int_{t_0}^{t_0+\theta} \mathfrak{J}[\phi](t_0, l_0, h_0, c_0, \pi_0)\right] + E\left[\int_{t_0}^{t_0+\theta} h(s)ds\right] \leq 0,$$

with $h(s) = O(s)$. Dividing by $E[t_0 + \theta]$ and taking the limit $n \to \infty$ (so $\theta \to 0$ and $E\left[\int_{t_0}^{t_0+\theta} h(s)ds\right] \to 0$) we get (3.24) as (π_0, c_0) can be arbitrary admissible pair.

The second part of the proof is to show that $V(t, l, h)$ is a subsolution as well. However, the proof of the second part of Theorem 4.1 in [21] can be applied verbatim here so we omit further details. •

□

The third result that is needed to finalize the proof of Theorem 1 is a *comparison principle* formulated below as Theorem 2. Results of this type are well-known in general for bounded controls, but due to the unboundned controls, classical proofs require adaptations for our case.

Theorem 2. *(***Comparison Principle***) Let $u(t, l, h)$ be an upper-semicontinuous concave viscosity subsolution of (3.11) on D and $V(t, l, h)$ is a supersolution of*

(3.11) on D which is bounded from below, uniformly continuous on D, and locally Lipschitz in D, such that $u(t,l,h) \to 0$, $V(t,l,h) \to 0$ as $t \to \infty$ and $|u(t,l,h)| + |V(t,l,h)| \leq O(|l|^\gamma + |h|^\gamma)$ for large l, h, where $0 < \gamma < 1$, uniformly in t. Then $u \leq v$ on $\overline{D}$.

Proof. Let us introduce $x := (l,h)$, $x \in \mathbb{R}^+ \times \mathbb{R}^+$ to make formulae shorter. Assume for contradiction that $\sup_{(t,x)\in\overline{D}}[u(t,x) - v(t,x)] > 0$. Let $T_n \to \infty$ be an increasing sequence of time moments, $m > 0$ be a parameter and

$$\Psi^{m,n}(t,x) = u(t,x) - v(t,x) - m(T_n - t).$$

Since $u, v \to 0$ as $t \to \infty$, for sufficiently large n and sufficiently small m the maximum of $\Psi^{m,n}$ must occur in an internal point of D. So let us assume that $\bar{m} > 0$ and T_n are such that $\sup_{(x,t)\in\overline{D}} \Psi^{\bar{m},n}(x,t)$ occurs in some point (t_0, x_0) with $t_0 < T_n$. Let us define two functions $\tilde{u}$ and ϕ

$$\begin{aligned} \tilde{u}(t,x) &= u(t,x) - \bar{m}(T_n - t), \\ \phi(t,x,y) &= \left|\frac{y-x}{\xi} - 4\varpi\right|^4 + \theta(l_x + h_x)^\lambda + \bar{m}(T_n - t), \end{aligned}$$

where $x = (l_x, h_x)$, $y = (l_y, h_y)$ and $\lambda \in (\gamma, 1)$, $\theta, \xi > 0$, $\varpi \in \mathbb{R}^2_+$ being parameters to be varied later. Finally, we look at the point $(\bar{x}, \bar{y}, \bar{t})$ where the following function achieves a maximum

$$\psi(t,x,y) = \tilde{u}(t,x) - v(t,y) - \phi(t,x,y).$$

Since $\bar{t}$ is an interior point we can write

$$2\bar{m} = u_t(\bar{t}, \bar{x}) - v_t(\bar{t}, \bar{y}). \tag{3.25}$$

On the other hand, one can bound $u_t(\bar{t}, \bar{x}, \bar{t}) - v_t(\bar{y})$ merely by ϕ and its derivatives which can be written down explicitly. It appears then, that as $\theta, \xi, \|\varpi\| \to 0$ the distance $\|\bar{x} - \bar{y}\|$ tends to zero and both $(\bar{t}, \bar{x})$, $(\bar{t}, \bar{y})$ are close to (t_0, x_0), so in the limit in terms of $\|\bar{x} - \bar{y}\| \to 0$ (3.25) leads to $\bar{m} \leq 0$ and we get a contradiction.

• $\square$

3.4 Homothetic reduction for logarithmic utility function

Though the HJB equation (3.11) generally fails to have a reduction with respect to the time variable, it is possible to reduce its dimension if the utility function is of the HARA type. In this chapter we work just with the logarithmic utility function. First of all, the logarithmic case allows one to consider time distributions with subexponential tails, while enjoying the homothetic reduction available for utility functions of the general HARA type. Secondly, the logarithmic case could in some sense be regarded as a limiting of the HARA case with γ tending to zero. This allows to translate all the obtained results to the general power case of HARA utility with only straightforward modifications. This correspondence is briefly mentioned in the literature, see [13] or [14], in [54] this correspondence is demonstrated but is used in a different framework. Further in this work we show that the logarithmic utility could not be only regarded as a formal limit of HARA utility, written in a special form, but also that this correspondence goes way deeper and that corresponding HJB equations are also connected under the same limit procedure as well as the algebraic structures that are admitted by these equations. This fact is new to our knowledge.

We study all possible symmetry reductions for optimization problem with HARA and logarithmic utility later in Chapter 5, Chapter 6 and Chapter 7. We also show there the connection between HARA and logarithmic utilities and corresponding optimization problems. In [9] such analysis was carried out for a model of illiquidity with frictions. The complete analysis for the current model with logarithmic and general HARA-type utility is done in this work later.

Rewriting the HJB equation (3.11) for the logarithmic utility function $U(c(t)) = \log c(t)$ we get

$$\begin{aligned} V_t(t,l,h) &+ \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ &+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0 \qquad (3.26) \\ G[\pi] &= \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h \\ &+ \pi(\alpha - r)V_l(t,l,h), \qquad (3.27) \\ H[c] &= -cV_l(t,l,h) + \overline{\Phi}(t)\log(c). \qquad (3.28) \end{aligned}$$

3. PORTFOLIO OPTIMIZATION IN THE CASE OF AN ILLIQUID ASSET WITH A GIVEN LIQUIDATION TIME DISTRIBUTION. GENERAL CASE.

Using the homothetic properties the logarithm and homogeneity of the differential operator applied to the value function in (3.26) we rewrite $V(t,l,h)$ in the following way

$$V(t,l,h) = W(t,z) - \Psi_1(t)\log h + \Psi_2(t), \tag{3.29}$$

having $z = l/h$ and $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ and $\Psi_2(t)$ to be chosen later.

The Hamiltonian terms $\max_\pi G[\pi]$ and $\max_c H[c]$ in (3.26) now become

$$\begin{aligned}\max_\pi G[\pi] &= \max_{\pi'=\pi/h}\left[\frac{1}{2}W_{zz}\sigma^2\pi'^2 + \pi'\left(-\eta\rho\sigma(W_z + zW_{zz}) + (\alpha-r)W_z\right)\right],\\ \max_c H[c] &= \max_{c'=c/h}[-c'W_z + \overline{\Phi}(t)\log(c')] + \overline{\Phi}(t)\log(h),\end{aligned} \tag{3.30}$$

and the candidates optimal policies after formal maximization are

$$\begin{aligned}\pi_\star(l,h) &= h\sigma^{-2}\left(\eta\rho\sigma z - ((\alpha-r)-\eta\rho\sigma)\frac{W_z}{W_{zz}}\right),\\ c_\star(l,h) &= h\frac{\overline{\Phi}(t)}{W_z}.\end{aligned} \tag{3.31}$$

We rewrite (3.26) using formulae (3.30)

$$\begin{aligned}& W_t + \Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\Psi_1(t) + \frac{\eta^2}{2}z^2W_{zz}\\ +\;& (\eta^2 + r - (\mu-\delta))zW_z + \delta W_z\\ +\;& \max_{\pi'}\left[\frac{1}{2}W_{zz}\sigma^2\pi'^2 + \pi'\left(-\eta\rho\sigma(W_z + zW_{zz}) + (\alpha-r)W_z\right)\right]\\ +\;& \max_{c'\geq 0}\left[-c'W_z + \overline{\Phi}(t)\log(c')\right] = 0.\end{aligned}$$

We provide the formal maximization of $H[\pi]$ and $G[c]$ and obtain

$$\begin{aligned}\max_\pi H[\pi] &= -\frac{1}{2}((\eta\rho - (\alpha-r)/\sigma)^2\frac{W_z^2}{W_{zz}} + 2\eta\rho(\eta\rho - (\alpha-r)/\sigma)zW_z + \eta\rho^2z^2W_{zz})\\ \max_c G[c] &= \overline{\Phi}(t)\left(\log\overline{\Phi}(t) - 1\right) - \overline{\Phi}(t)\log W_z.\end{aligned}$$

so (3.26) becomes

$$\begin{aligned}& W_t + \Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu-\delta)\right)\Psi_1(t) + \overline{\Phi}(t)(\log\overline{\Phi}(t) - 1) + d_2z^2W_{zz}\\ -\;& \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} + d_3zW_z + \delta W_z - \overline{\Phi}(t)\log W_z = 0,\end{aligned} \tag{3.32}$$

where

$$
\begin{aligned}
d_1 &= \frac{\alpha - r - \eta\rho\sigma}{\sigma^2}, \quad d_2 = \frac{1}{2}\eta^2(1-\rho^2), \\
d_3 &= 2d_2 + \frac{\rho\eta}{\sigma}(\alpha - r) + r - (\mu - \delta). \qquad (3.33)
\end{aligned}
$$

Now by choosing $\Psi_2(t)$ as a solution of the equation

$$
\Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu - \delta)\right)\Psi_1(t) + \overline{\Phi}(t)(\log \overline{\Phi}(t) - 1) = 0, \quad \Psi_2(t) \to 0, t \to \infty,
$$

we can cancel out the terms dependent only on t in the equation (3.32). We arrive at

$$
W_t - \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} + d_2 z^2 W_{zz} + d_3 z W_z + \delta W_z - \overline{\Phi}(t)\log W_z = 0. \qquad (3.34)
$$

The two dimensional PDE (3.34) can be obtained in few steps using the symmetry properties of (3.26) - (3.28).

3.5 Bounds for the value function

The main tool we are going to use to obtain the bounds is the comparison principle given by Theorem 2. Since (3.34) is a two-dimensional PDE and by itself is not a HJB equation, we argue as follows. Any formal sub- or super- solution of (3.34) can be transformed to a sub- or super- solution of (3.26) with a substitution described by (3.29). On the other hand, for the HJB equation (3.26) Theorem 1 and Theorem 2 hold and we can obtain a lower and upper bound. In order to comply with the Definition 1 we have to take the equation (3.34) with the minus sign.

Determining an upper bound demands specific information on the cumulative distribution function $\Phi(t)$ of the liquidation time. In the next Section this issue is addressed specifically for two practically applicable cases of exponentially and Weibull distributed liquidation time T.

Remark 5. *We choose Weibull distribution as an illustration for a non-exponential case, since it is good in descriptions of survival data as it is explained in [44] and [17], and has a local maximum that can be easily interpreted from the financial point of view. For a number of financial assets one can estimate a standard* *time-lag between the placement of an offer and a moment of actual sale (see, of instance, [7]). This lag plays a very important role in empirical studies such as [12] and can, for example, be attributed to the local maximum of liquidation time distribution.*

A lower bound, however, could be found without any specific information on $\Phi(t)$. Let us look on an optimal consumption problem without random income. This is a classical two dimensional Merton's problem for which we can write the HJB equation on the value function $u(t, z)$. This problem corresponds to (3.26) but without any terms, containing the derivatives with respect to h and with a notation $V \to u$, $l \to z$

$$u_t + rlu_z + \max_{\pi} G[\pi] + \max_{c \geq 0} H[c] = 0, \tag{3.35}$$

$$G[\pi] = \frac{1}{2} u_{zz}(t, z)\pi^2\sigma^2 + \pi(\alpha - r)u_z(t, z),$$

$$H[c] = -cu_z(t, z) + \overline{\Phi}(t) \log(c). \tag{3.36}$$

After the formal maximization, one gets

$$u_t + rlu_z - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \frac{u_z^2}{u_{zz}} + \overline{\Phi}(t)\left(\log \overline{\Phi}(t) - \overline{\Phi}(t)\right) - \overline{\Phi}(t) \log u_z = 0.$$

We look for a solution in the form $u(t, z) = \Psi_1(t) \log z + \Theta_1(t)$, where again $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ and $\Theta_1(t)$ is a solution of

$$\Theta_1' + \Psi_1 \left(r + \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2}\right) - \overline{\Phi}(\overline{\Phi} - \log \overline{\Phi} + \log \Psi_1) = 0. \tag{3.37}$$

One can easily check that such u tends to zero uniformly as $t \to \infty$ and since the solution of (3.35) is a lower bound for our three-dimensional problem we obtain the following inequality for the lower bound

$$\Psi_1(t) \log z + \Theta_1(t) \leq W(z, t) = V(t, l, h) - \Psi_1 \log h + \Psi_2(t),$$

or

$$\Psi_1(t) \log l + \Theta_1(t) - \Psi_2(t) \leq V(t, l, h).$$

3.6 Results of the chapter and outlook

In this Chapter we have formulated the optimization problem for a portfolio that consists of a riskless liquid, risky liquid and risky illiquid assets with an exogenous random liquidation time. We have shown that one can regard a problem with infinite time horizon and special weight function that is characterized by the probability distribution of the liquidation time T instead of a problem with an exogenous random liquidation time. With a help of viscosity solution technique that was also introduced in this chapter we proved the existence and uniqueness of the solution for such infinite time horizon problem with special weight function that depends on distribution of T. For any probability distribution for which a unique solution exists a lower bound for the value function was found.

In the next Chapter we consider specific liquidation time distributions. First we take the most simple one - an exponential distribution. Later, in Chapter 5, we prove that the case of exponential distribution is the only exceptional case, where the admitted Lie group is richer. We get asymptotically tight bounds for the value function and derivatives, which lead to asymptotic formulae for the optimal policies. Not surprisingly, in the limit case when the random income vanishes the value function and optimal policies coincide with the classical Merton solution for the logarithmic case. Another somewhat more complicated case is the Weibull distribution is regarded in Chapter 4 as well. In this case the bounds have no elementary representation, but their asymptotic can be derived using incomplete gamma functions.

4

Portfolio optimization in the case of an illiquid asset with an exponential or Weibull liquidation time distribution

In this chapter we develop the results obtained in the previous Chapter and compare two different liquidation time distributions. The crucial difference between the exponentially distributed and Weibull distributed liquidation time is that a the latter can have a local maximum that could be associated with an average time-lag between the offer and the sale.

4.1 The case of exponential distributed liquidation time and logarithmic utility function

Now we examine the optimal consumption problem introduced before in Chapter 3 in the case of the logarithmic utility. In Chapter 3 we have established that the optimal strategy does exist and the value function is the viscosity solution of the HJB equation, now it is desirable to have the optimal policy in the feedback form (3.31). In a general situation the feedback optimal policy is hard to establish since the value function is not a priori smooth. On the other hand, smoothness of the value function simplifies the problem so it becomes amenable to standard verification theorems of optimization theory, see e.g. [25]. Here we prove that in the case at hand the value function is twice differentiable. As far as we know this fact was not explicitly addressed before, though the structure of our proof is similar to the paper [22] where the smoothness was proved for the HARA utility case. Since the case without stochastic income is known to have a closed form solution and was derived by Merton [42], it is plausible to consider it as a zero-term approximation. Keeping that in mind, we will rigorously prove that value function tends to the Merton closed form solution in the limit of vanishing random income. Recall the definition of the value function

$$V(t,l,h) = \max_{(\pi,c)} E\left[\int_t^\infty e^{-\kappa t}\log(c)dt|L(t)=l, H(t)=h\right], \quad \kappa>0. \tag{4.1}$$

At first let us note that in the exponential liquidation time distribution case the problem is homogenous in time. We introduce $\tilde{V}(l,h)$

$$\tilde{V}(l,h) = \max_{(\pi,c)} E\left[\int_t^\infty e^{-\kappa(s-t)}\log(c)ds\right] = \max_{(\pi,c)} E\left[\int_0^\infty e^{-\kappa v}\log(c)dv\right],$$

which is independent on time. Substituting

$$V(t,l,h) = e^{-\kappa t}\tilde{V}(l,h)$$

into the HJB equation (3.11) we arrive at a time-independent PDE on $\tilde{V}(l,h)$. With a slight abuse of notation, hereafter we will use the same letter V for $\tilde{V}$. The reduced equation takes the form

$$\begin{aligned}
& \frac{1}{2}\eta^2 h^2 V_{hh}(l,h) + (rl+\delta h)V_l(l,h) + (\mu-\delta)hV_h(l,h) + \max_\pi G[\pi] + \max_{c\geq 0} H[c] \\
= \; & \kappa V(l,h), \\
& G[\pi] = \frac{1}{2}V_{ll}(l,h)\pi^2\sigma^2 + V_{lh}(l,h)\eta\rho\pi\sigma h + \pi(\alpha-r)V_l(l,h), \qquad (4.2)\\
& H[c] = -cV_l(l,h) + \log(c). \qquad (4.3)
\end{aligned}$$

Now using substitution (3.29) with $\Psi_1 = \frac{1}{\kappa}$ and $\Psi_2 = \frac{1}{\kappa^2}(\mu-\delta-\frac{\eta^2}{2})$ we can argue exactly as in the general case and represent $V(l,h)$ in the form

$$V(l,h) = v(z) + \frac{\log h}{\kappa} + \frac{1}{\kappa^2}\left(\mu-\delta-\frac{\eta^2}{2}\right), \quad z = l/h, \tag{4.4}$$

so $v(z)$ satisfies the ordinary differential equation of second order

$$\begin{aligned}\frac{\eta^2}{2}z^2v'' &+ \max_{\pi}\left[\frac{1}{2}\pi^2\sigma^2v' - \pi\left((v'+zv'')\eta\rho\sigma + (\alpha-r)v'\right)\right] \\ &+ \max_{c\geq-\delta}\left[-cv_z + \log(c+\delta)\right] = \kappa v, \qquad (4.5)\end{aligned}$$

where $v' = v_z$ and the dimension of the problem is reduced to one. It is important to note that such reduction was possible due to the exponential decay, the homothetic proprety of the logarithmic function and the linearity of the control equations, which make the reduction (4.4) sound.

Assuming that v is smooth and strictly concave, we perform a formal maximization of the quadratic part (4.2) which leads to

$$\kappa v v'' = -\frac{d_1^2}{2}(v')^2 + d_2z^2(v'')^2 + d_3zv'v'' - v''\left[1+\log(v')\right], \qquad (4.6)$$

where again d_1, d_2 and d_3 are defined in (3.33).

Coming back to the original variables we obtain the optimal policies in the form

$$c_\star(l,h) = \frac{h}{v'(l/h)}, \quad \pi_\star(l,h) = -\frac{\eta\rho}{\sigma}l - h\frac{d_1}{\sigma}\frac{v'(l/h)}{v''(l/h)}. \qquad (4.7)$$

Summing up, we announce the main result of this Chapter.

Theorem 3. *Suppose $r-(\mu-\delta) > 0$ and $d_1 = \frac{\alpha-r-\eta\rho\sigma}{\sigma^2} \neq 0$.*

- *There is the unique $C^2(0,+\infty)$ solution $v(z)$ of (4.6) in a class of concave functions.*

- *For $l,h>0$ the value function is given by (4.4). For $h=0, l>0$ the value function $V(l,0)$ coincides with the classical Merton solution*

$$V(l,0) = \frac{1}{\kappa^2}\left[r + \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2} - \kappa\right] + \frac{\log(\kappa l)}{\kappa} = M + \frac{\log(\kappa l)}{\kappa}. \qquad (4.8)$$

- *If the ratio between the stochastic income and the total wealth tends to zero, the policies $(\pi^\star, c^\star)$ given by (4.7) tend to the classical Merton's policies*

$$c_\star(l,0) \sim \kappa l, \ \pi_\star(l,0) \sim -\frac{(\alpha-r)l}{\sigma^2}\frac{V_l^2}{V_{ll}}. \qquad (4.9)$$

- *Policies (4.7) are optimal.*

We have shown that the solution exists and tends to Merton case when $h=0$. In the next step we will show the smoothness of the solution.

4.1.1 The dual optimization problem and smoothness of the viscosity solution

In this Section we introduce the dual optimization problem with a synthetic asset such that the optimization equation formally coincides with (4.6). The regularity of the dual problem proves the regularity of the original one due to the uniqueness of the viscosity solution.

Consider the investment-consumption problem with the wealth process Z_t defined by

$$\begin{aligned} Z_t &= (d_3 Z_t + d_1 \sigma \pi_t - c_t)dt + \sigma \pi_t dW_t^1 + \eta Z_t \sqrt{1-\rho^2} dW_t^2, \\ Z_0 &= z \geq 0, \end{aligned} \tag{4.10}$$

where d_1 and d_3 are defined in (3.33). We define the set of admissible controls $\hat{\mathcal{A}}(z)$ as the set of pairs (π, c) such that there exists an a.s. positive solution Z_t of the stochastic differential equation (4.10), $c_t \geq -\delta$ and c and π satisfy the integrability conditions (3.3).

The investor wants to maximize the average utility given by

$$\hat{\mathfrak{U}}(c) = E\left[\int_0^\infty e^{-\kappa\tau} \log(\delta + c(\tau)) d\tau\right]$$

and the value function w is defined as $w(z) = \sup_{(\pi,c)\in\hat{\mathcal{A}}(z)} \hat{\mathfrak{U}}(c)$.

The associated HJB equation is reduced to the ODE

$$\begin{aligned} \kappa w &= d_2 z^2 w'' + \max_\pi \left[\frac{1}{2}\sigma^2\pi^2 w'' + d_1 \sigma \pi w'\right] + d_3 z w' \\ &+ \max_{c\geq -\delta} \left[-cw' + \log(c+\delta)\right], \end{aligned} \tag{4.11}$$

Next, keeping in mind $w' > 0, w'' < 0$, we can rewrite (4.11) as

$$-\frac{d_1^2}{2}\frac{(w')^2}{w''} + d_2 z^2 w'' + d_3 z w' + \delta w' - 1 - \log w' - \kappa w = 0. \tag{4.12}$$

Now, it is easy to see that (4.11) reduces to (4.6) assuming that w is smooth. Thus, if we prove that w is smooth and concave, we will get the desired result for v in (4.5) as well. The possibility to switch back and forth from V in (4.3) to v in (4.5) and w in (4.11) is guaranteed by the existence and uniqueness of the viscosity solutions given by Theorem 1. On the other hand, if a function is the value function for the corresponding optimization problem, and the HJB equations formally coincide, the value functions must coincide as well due to uniqueness. Therefore, it is sufficient to prove that w is smooth.

From the previous Chapter we already know that if $D = (0, \infty)$ the following theorem holds.

Theorem 4. *The function w is the unique viscosity solution of (4.11) in D. And the value function $V(l, h)$ is the unique viscosity solution of (4.4) in $D \times D$.*

Let us now prove the smoothness of the solution and of its' first derivative.

Theorem 5. *The function w in (4.12) is the unique concave $C^2(D)$ solution of (4.11).*

To start with the proof of the theorem we need some explicit bounds for w.

Lemma 3. *The following bounds hold for $w(z)$*

$$C_1 \log(z + C_2) < w(z) < (z + C_3)^\gamma, \quad z \in \Omega \tag{4.13}$$

for some constants $C_1, C_2, C_3 > 0$ and $0 < \gamma < 1$.

Proof. The function

$$W^-(z) = C_1 \log(z + C_2), \quad z \in \Omega$$

is a subsolution for (4.12) as the coefficient of the leading logarithmic term is negative provided $C_1, C_2 > 0$ are appropriately chosen. On the other hand, the function

$$W^+(z) = (z + C_3)^\gamma, \quad z \in \Omega$$

is a supersolution provided $0 < \gamma < 1$ is sufficiently close to 1. Indeed, the leading term is z^γ with the coefficient $-(d_1^2(w')^2)/(2w'')$, which in turn grows as $-\gamma/(\gamma - 1)$ and becomes arbitrarily large as γ tends to 1.

Thus, the desired bound (4.13) is a consequence of the comparison principle formulated in Theorem 2. • □

Now we can prove Theorem 5.

Proof. It is known that uniformly elliptic equations enjoy regularity, but as before the main obstacle is the lack of uniform bounds. The main idea of the proof is to approximate the original problem with a convergent family of optimization problems such that the approximating equations is uniformly elliptic and thus smooth. Then the smoothness follows from the stability of viscosity solutions and uniqueness.

Step 1. Consider the value function $w_L(z) = \sup_{(\pi,c)\in\hat{\mathcal{A}}(z)} \hat{\mathfrak{U}}(c)$ for the problem with the additional strategy constraint $-L \le \pi_t \le L$ for almost every t. Arguing as in Section 3.1 we conclude that that w_L is an increasing continuous function, which is the unique viscosity solution to

$$\begin{aligned} \kappa w_L &= d_2 z^2 w_L'' + \max_{-L\le\pi\le L}\left[\frac{1}{2}\sigma^2\pi^2 w_L'' + d_1\pi w_L'\right] + d_3 z w_L' \\ &+ \max_{c\ge-\delta}\left[-c w_L' + \log(c+\delta)\right]. \end{aligned} \tag{4.14}$$

Moreover, the bounds of Lemma 3 hold so $C_1\log(z+C_2) < w^L(z) < (z+C_3)^\gamma$.

Thus, there exists a concave function $\hat{w}$ such that $w_L \to \hat{w}, L \to \infty$ locally uniformly. Then due to the stability property and uniqueness of the viscosity solution the function $\hat{w}$ is a viscosity solution of (4.11) and thus coincides with w. Therefore $w_L \to w, L \to \infty$ locally uniformly.

Step 2. We claim that w_L is a smooth function on an arbitrary interval $[z_1, z_2]$ such that $z_1 > 0$. Due to concavity we may assume that derivatives $w_L'(z_1), w_L'(z_2)$ exist. On the one hand the function w_L is the unique solution of the boundary problem

$$\begin{aligned} \kappa u &= d_2 z^2 u'' + \max_{-L\le\pi\le L}\left[\frac{1}{2}\sigma^2\pi^2 u'' + d_1\sigma\pi u'\right] + d_3 z u' \\ &+ \max_{c\ge-\delta}\left[-c u' + \log(c+\delta)\right], \\ u(z_1) &= w_L(z_1), \quad u(z_2) = w_L(z_2), \quad z\in[z_1,z_2]. \end{aligned} \tag{4.15}$$

On the other hand, according to the general theory of fully nonlinear elliptic equations of second order of Bellman type in a compact region, (see *Krylov* [37]), (4.14) has a unique C^2 solution in $[z_1, z_2]$ that coincides with w_L and w_L is smooth on $[z_1, z_2]$.

Step 3. We show that the constraint $-L \le \pi_t \le L$ is superfluous for sufficiently large L and can be eliminated. First it is clear that due to concavity and monotonicity of w_L, the condition $-L \leq \pi_t \leq L$ in (4.14) can be substituted with $\pi_t \leq L$. Now we prove that

$$\sup_{z\in(z_1,z_2)} \left[-\frac{d_1^2(w_L')^2}{2w_L''}\right] < L$$

for sufficiently large L. Assume the contrary for contradiction. Then there is a sequence $z_n \in (z_1, z_2)$, $L_n \to \infty$ such that

$$\begin{aligned} &- \frac{d_1^2(w_L'(z_n))^2}{2w_L''(z_n)} > L_n, \\ \kappa w_L \quad &\geq \quad d_2 z^2 w_L'' - L_n + d_3 z w_L' + [\delta w_L' - 1 - \log w_L']\,. \end{aligned} \tag{4.16}$$

Since $w_L \to w$ and both function are monotone and concave there exist constants C_1, C_2 such that $C_1 < w_L'(z) < C_2, \quad z \in [z_1, z_2]$ for all sufficiently large L, and also $w_L'' \to 0$ as $n \to \infty$. But this contradicts (4.16) as z_n takes values in a bounded interval so $w_L(z_n)$ is bounded as well.

Step 4. We are going to show that there is a constant $K < 0$ which does not depend on L such that $w_L''(z) < K, \quad z \in [z_1, z_2]$. Arguing again by contradiction suppose there is a sequence $z_n \in [z_1, z_2]$ such that $w_L''(z_n) \to \infty$. Then analogously to Step 3, the right hand side of (4.15) grows to infinity since $w_L'(z)$ on the interval that is bounded. At the same time the left hand side stays bounded as a value of a continuous function on a bounded interval.

Step 5. Putting it all together, we have the following chain of implications. The functions w_L are unique smooth solutions in the class of concave functions to the boundary problem (4.15) for some sufficiently large $L > 0$. Since $w_L \to w$, it follows that w is the unique viscosity solution of (4.15) in the class of concave functions. On the other hand, the equation (4.15) possesses the unique smooth solution, see [37], which must coincide with the viscosity solution. Thus w is a C^2-smooth function on $[z_1, z_2]$ and the claim of the theorem follows since the interval is arbitrary. • □

4.1.2 Asymptotic behavior of the value function

In this Section we examine the asymptotic behavior of the value function $V(t, l, h)$ in (4.3) and show that as $l/h \to \infty$ it becomes the classical Merton solution.

Theorem 6. *There is a positive constant C_1 such that*

$$\begin{aligned} M &+ \frac{\log(\kappa l)}{\kappa} \leq V(l,h) \leq M + \frac{\log(\kappa(l+C_1\delta h))}{\kappa}, \\ M &= \frac{1}{\kappa^2}\left[r + \frac{1}{2}\frac{(\alpha-r)^2}{\sigma^2} - \kappa\right] \end{aligned}$$

where M is a constant from the Merton's formula (4.8).

Proof. The proof is based on the idea mentioned in Lemma 1, but in the specific exponentially distributed liquidation time case the bounds could be found explicitly. The left-hand inequality is obvious since any strategy (π, c) for the classical problem with $L_0 = l, H_0 = 0$ is admissible for the problem with any non-zero initial endowment as well. For the right-hand side, let us consider a fictitious investment-consumption problem without any stochastic income but with an additional synthetic asset with the price process S': $dS'_t = \alpha' S'_t + \sigma'_1 S'_t dW_t, \quad t \geq 0$, $S'_0 = s', \quad s' > 0$ with appropriate constants α' and σ'. Next, we define the *initial wealth equivalent* of the stochastic income defined by

$$\begin{aligned} V_\delta(l,h) &= \delta E_h\left[\int_0^\infty e^{-rt}\xi_t H_t dt\right], \\ \xi_t &= \exp\left(-\frac{1}{2}(\theta_1^2+\theta_2^2) + \theta_1 W_t^{(1)} + \theta_2 W_t\right), \end{aligned}$$

where $\theta_1 = (\alpha - r)/\sigma_1$ and $\theta_2 = (\alpha' - r)/\sigma'_1$.

As we mention in the proof of Lemma 1, by a careful choice of the constants α', σ' the stochastic income rate H_t can be replicated by a self-financing strategy on the complete market (B_t, S_t, S'_t) with the additional initial endowment $f(h) < C_1\delta h$, for similar reasoning see [35], [32] and [21]. Thus, any average utility generated by the strategy $(\pi, c) \in \mathcal{A}(l,h)$ can be attained in the settings of a classical Merton's problem with the initial wealth $l + f(h) < l + C_1\delta h$. This actually gives the right-hand bound in Theorem 1. • □

From this theorem we immediately get that $V(l,h)$ behaves as the classical Merton solution (4.8) as $\delta \to 0$ or $l/h \to \infty$.

Corollary 1. *$V_\delta(l,h)$ converges locally uniformly to $M + \log(\kappa l)/\kappa$ as $\delta \to 0$.*

Corollary 2. *$V(l,h) = M + \log(\kappa l)/\kappa + O(1/z)$ as $z = l/h \to \infty$. Also for the function $w(z)$ we obtain $w(z) = (M - K) + \frac{\log(\kappa z)}{\kappa} + O(1/z)$.*

Proof. Indeed,

$$\left|V(l,h) - M - \frac{\log(\kappa l)}{\kappa}\right| < \left|\frac{1}{\kappa}\left(\log(\kappa(l+\delta C_1 h)) - \log(\kappa l)\right)\right| = O\left(\frac{1}{z}\right). \quad (4.17)$$

The formula immediately follows from the form of $V(l,h)$.• □

Finally, we verify that the optimal policies given by (4.7) asymptotically give the Merton strategy (4.9).

Lemma 4. *For the value function $w(z)$ holds*

$$w'(z) = \frac{1}{\kappa z} + o\left(\frac{1}{z}\right), \quad z \to \infty. \quad (4.18)$$

Proof. Consider the function w_λ defined as

$$w_\lambda(z) = w(\lambda z) - \frac{\log(\lambda)}{\kappa},$$

so that w_λ solves (4.11) but with the term

$$F(w_z) = \max_{c \geq -\delta}\left[-c w_z + \log(c+\delta)\right]$$

replaced by

$$F_\lambda(w_z) = \max_{c \geq -\delta/\lambda}\left[-c w_z + \log(c + \frac{\delta}{\lambda})\right].$$

Then, by Corollary 1 w_λ converges locally uniformly to the Merton's value function

$$v(z) = (M-K) + \frac{\log(\kappa z)}{\kappa}.$$

We note that v solves (4.11) with $\delta = 0$ that is delivered by $F_\infty(\cdot) = \lim_{\lambda\to\infty} F_\lambda(\cdot)$. Thus, since w_λ is concave, the uniform convergence of w_λ to v implies the convergence of derivatives, so $\lim_{\lambda\to\infty} w'_\lambda(z) = v'(z) = \frac{1}{\kappa z}$. Hence,

$$\lim_{\lambda\to\infty} w'_\lambda(1) = \lim_{\lambda\to\infty} \lambda v'(\lambda) = \frac{1}{\kappa},$$

which proves the lemma. • □

Theorem 7. *The following asymptotic formulae hold for the optimal policies (4.7) as $z = l/h \to \infty$.*

$$\frac{c^\star}{l} \sim \frac{1}{\kappa}, \quad \frac{\pi^\star}{l} \sim \frac{\alpha - r}{\sigma^2}. \quad (4.19)$$

Proof. The formula for $c^\star$ in (4.19) immediately follows from Lemma 4. For the second part, we rewrite (4.7) in a form

$$\frac{\pi^\star}{l} = \frac{\eta\rho}{\sigma} - \frac{k_1}{\sigma^2}\frac{zv'(z)}{z^2v''(z)}.$$

To calculate the limit value of $z^2v''(z)$ we rewrite (4.6) as a quadratic equation with respect to v_{zz}. Since $v_{zz} < 0$ we choose the negative root and obtain

$$v_{zz}(z) = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

where

$$\begin{aligned} A &= \frac{1}{2}\eta^2(1-\rho)^2z^2, \\ B &= k(zv_z) - 1 - (M - C)\kappa + o(1), \\ C &= -\frac{k_1^2}{2\sigma^2}(v_z)^2. \end{aligned}$$

Expanding all constants and using $zw_z = 1/\kappa + o(1)$ we finally get

$$z^2v_{zz}(z) = \frac{(\alpha - r)l}{\sigma^2} + o(1).\bullet$$

□

The facts that the solution exists, is unique and smooth give an opportunity for numerical calculations. For example, basing on a script, developed by *Andersson, Svensson, Karlsson and Elias*, see [3], with some modifications and corrections of minor mistakes we can obtain the solution for the exponential case and compare it with a two-dimensional Merton solution as shown on the Figure 4.1.

4.2 The case of a Weibull distributed liquidation time and a logarithmic utility function

One of the most natural ways to extend the framework of a randomly distributed liquidation time described in Section 3.1 is to introduce a distribution with a probability density function that has a local maximum unlike exponential distribution. It is very natural to expect that the assets of a certain type might have a time-lag between the moment when the sell offer is opened and a time when someone reacts on it. From the practitioner's point of view an empirical estimation of such time-lag is a natural measure of illiquidity that can give an insight into the strategy of a portfolio management. In this Section we look closely on a Weibull distribution that has a local maximum. The Weibull distribution is commonly used in survival analysis, in reliability engineering and failure analysis, and in industrial engineering to describe manufacturing and delivery times. It seems to be quite adequate for the studied case. We demonstrate that the proposed framework is applicable for this case, show the existence and uniqueness of the solution and using a numerical algorithm generate an insight into how this case differs from the exponential illiquid and Merton's absolutely liquid cases.

In this Section we will discuss the case when the liquidation time T is a random Weibull-distributed variable independent of the Brownian motions (W^1, W^2). The probability density function of the Weibull distribution is defined as follows

$$\phi(x,\lambda,k) \quad = \quad \begin{cases} \frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1} e^{-(t/\lambda)^k}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

where $\lambda > 0$ and $k, \lambda - const.$

Let us also introduce as before the cumulative distribution function

$$\Phi(x,\lambda,k) \quad = \quad \begin{cases} 1 - e^{-(t/\lambda)^k}, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0, \end{cases}$$

and a survival function $\overline{\Phi}(t) = 1 - \Phi(t)$. We will often omit the constant parameters λ and k in notations for shortness.

It is important to notice that when $k = 1$ the Weibull-distribution turns into exponential one, that we have already discussed before and for $k > 1$ its probability density has a local maximum. This situation corresponds to our economical motivation.
The equation (3.11) is the same as before but the term that corresponds to $\overline{\Phi}$ is

naturally replaced by Weibull survival function

$$\begin{aligned}
V_t(t,l,h) &+ \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\
&+ \max_{\pi} G[\pi] + \max_{c\geq 0} H[c] = 0, \qquad (4.20) \\
G[\pi] &= \frac{1}{2}V_{ll}(t,l,h)\pi^2\sigma^2 + V_{lh}(t,l,h)\eta\rho\pi\sigma h \\
&+ \pi(\alpha-r)V_l(t,l,h), \qquad (4.21) \\
H[c] &= -cV_l(t,l,h) + e^{-(t/\lambda)^k}U(c), \qquad (4.22)
\end{aligned}$$

Proposition 2. *All the conditions of the Theorem 1 hold for the case of the Weibull distribution with $k > 1$ and, therefore, there exists a unique solution for the problem (4.20).*

Indeed the conditions 1., 3. and 4. are not altered since we work with the same logarithmic utility and one can easily see that the Weibull cumulative function satisfies the condition 2. for the case $k > 1$.

Analogously to the equation (3.34) one can obtain a two dimensional equation using a known reduction $z = l/h$. We study all the symmetry reductions of this model for the exponential and general case in the next chapters (see also [11]). Yet here let us just list a two dimensional equation that corresponds to the Weibull case

$$W_t - \frac{d_1^2}{2}\frac{(W_z)^2}{W_{zz}} + d_2 z^2 W_{zz} + d_3 z W_z + \delta W_z - e^{-(t/\lambda)^k}\log W_z = 0, \qquad (4.23)$$

where d_1, d_2 and d_3 correspond to the constants for the general case (3.33).

The function $\Psi_1(t) = \int_t^\infty \overline{\Phi}(s)ds$ can be defined explicitly as

$$\Psi_1(t) = \frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right),$$

where $\Gamma(\alpha, x)$ is an incomplete gamma function, see [1] for details. For this function we can use the series representation by Laguerre polynomials and asymptotic representation [1], [38].

The lower bound for $W(z,t)$ can be found exactly as in Section 3.5

$$W(z,t) = V(t,l,h) - \Psi_1 \log h - \Psi_2(t) \geq \Psi_1(t)\log z + (\Theta(t) - \Psi_2(t)),$$

where the behavior of the functions Ψ_1, Ψ_2 and Θ for $t \to \infty$ can be now well defined.

The equation (3.34) for the auxiliary function $\Psi_2'(t)$ takes the form

$$\Psi_2'(t) + \left(-\frac{\eta^2}{2} + (\mu - \delta)\right) \frac{\lambda}{k} \Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) - e^{-(t/\lambda)^k}((t/\lambda)^k + 1) = 0,$$
$$\Psi_2(t) \to 0, t \to \infty. \tag{4.24}$$

The solution for this equation can be found explicitly

$$\Psi_2(t) = -\left(-\frac{\eta^2}{2} + (\mu - \delta)\right) \frac{\lambda}{k} \Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) + e^{-\left(\frac{t}{\lambda}\right)^k} \left(\left(\frac{t}{\lambda}\right)^k + 1\right). \tag{4.25}$$

Equation (3.37) for Θ is now

$$\begin{aligned} \Theta' \quad &+ \quad \left(r + \frac{1}{2}\frac{(\alpha - r)^2}{\sigma^2}\right) \frac{\lambda}{k} \Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) - e^{-(t/\lambda)^k} \Big[e^{-(t/\lambda)^k} + (t/\lambda)^k \\ &+ \quad \log\lambda - \log k \log\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right)\Bigg] = 0. \end{aligned} \tag{4.26}$$

One can find an explicit solution for it as well

$$\begin{aligned} \Theta(t) \quad &= \quad -\left(r + \frac{(\alpha - r)^2}{2\sigma^2}\right) \frac{\lambda}{k} \Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right) \\ &+ \quad e^{-\left(\frac{t}{\lambda}\right)^k} \left(e^{-\left(\frac{t}{\lambda}\right)^k} + \left(\frac{t}{k}\right)^k + \ln\left[\frac{\lambda}{k}\Gamma\left(\frac{1}{k}, \left(\frac{t}{\lambda}\right)^k\right)\right]\right). \end{aligned}$$

Since $\frac{1}{k} > 0$ we can show that asymptotically as $t \to \infty$

$$\begin{aligned} \Psi_1(t) \quad &\to \quad \frac{\lambda^k}{k}(t)^{1-k} e^{-(t/\lambda)^k}\left(1 + O\left(t^{-k}\right)\right), \\ \Psi_2(t) \quad &\to \quad -\frac{1}{k} t e^{-(t/\lambda)^k}\left(1 + O\left(t^{-k}\right)\right), \quad k > 1, \\ \Theta(t) \quad &\to \quad \frac{\lambda - k}{\lambda k} t e^{-(t/\lambda)^k}\left(1 + \frac{(k-1)k\lambda}{\lambda - k} t^{-k}\ln t + O\left(t^{-k}\right)\right). \end{aligned}$$

It follows from the asymptotic behavior that the value function in (4.20) tends to zero faster than $e^{-\kappa t}$ and consequently Theorem 1 is applicable for the Weibull-distributed liquidation time.

On the Figure 4.1 one can see the results of the numerical simulation for consumption and investment strategies that we run for a Weibull and exponential case. As the parameter k that is responsible for the form of Weibull distribution increases the optimal policies differ significantly from the exponential liquidation time case. As z increases, i.e. the illiquid part of the portfolio becomes insufficiently small, we can see that all the policies tend to one solution which is, in fact, a Merton solution for a two-asset problem derived in [42].

It is interesting to note here, that as z becomes smaller (i.e. the illiquid part of the wealth becomes comparable with the liquid part) optimal policy tends to reduce the amount of liquid wealth invested in risky assets. This result goes in line with empirical results on house ownership, analyzed in [59].

It is especially important to note that the optimal policies significantly differ from Merton solution when illiquidity becomes higher. Already when an amount of illiquid asset is more than 5% of the portfolio value the percentage of capital that is not invested in a risky stock is higher than in Merton model.

4.3 Results of the Chapter

Applying Theorem 1 obtained in Chapter 3 to two different liquidation time distributions in the case of logarithmic utility function (i.e., exponential and Weibull distribution) we proved the smoothness of the solution and found a lower and upper bounds for the value function of a problem with an exponentially distributed random liquidation time. For the Weibull distributed liquidation time with parameter $k > 1$ we have demonstrated the applicability of a general Theorem 1 that proves the existence and uniqueness of a viscosity solution and also found a lower and upper bound for it. We have also demonstrated numerically that the resulting strategies for such portfolio differ from the Merton case yet tend to it when illiquidity becomes infinitely small.

Weibull distribution is regarded as a practically relevant an example of a non-exponential distribution that can be used in the proposed framework. This example shows how the developed approach to the illiquidity opens up a class of optimization problems that could be treated in a similar way. In the next chapter we carry out Lie group analysis of the obtained equations and show how algebraic structures standing behind them actually make exponential distribution

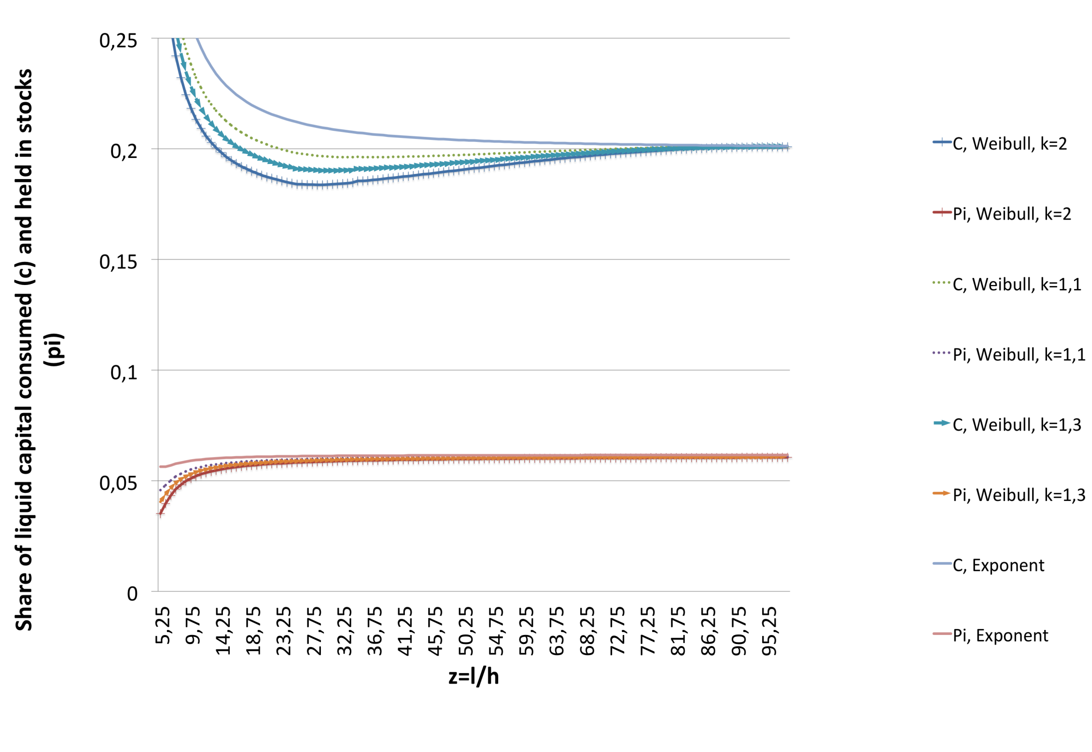

Figure 4.1: *Consumption stream c and the share of liquid capital π stored in a risky asset depending on the ratio between the liquid and illiquid asset. As illiquid asset value becomes infinitely small the policies tend to Merton policies for a two-asset problem. We used the following parameters for assets $r = 0.01, \sigma = 0.5, \delta = 0.02, \rho = 0.4, \mu = 0.05, \eta = 0.3$ and $\lambda = 2$.*
Der Verbrauchsanteil c und der Investitionsanteil π einer risikoreichen liquiden Position des liquiden Kapitals als Funktionen von dem Verhältniss zwischen den liquiden und illiquiden Posititionen. Wenn der Wert der illiquiden Position infinitesimal klein wird streben diese Strategien gegen Mertonsche Strategien für das Zwei-Position-Problem. Wir haben folgende Parameter bei der Positionen benutz $r = 0.01, \sigma = 0.5, \delta = 0.02, \rho = 0.4, \mu = 0.05, \eta = 0.3$ und $\lambda = 2$.

a distinguished case.

5

Lie group analysis of the problem

Study of optimization problems with an illiquid asset leads to three dimensional nonlinear Hamilton-Jacobi-Bellman (HJB) equations. Such equations are not only tedious for analytical methods but are also quite challenging form a numeric point of view. One of the standard techniques is to find an inner symmetry of the equation and reduce the number of variables at least to two or if possible to one. The problems of lower dimensions are usually better studied and are, therefore, easier to handle.

All papers known to us devoted to three dimensional HJB equations provide variable substitutions without any remark on how to get similar substitution in other cases or why they use this or that substitution. Yet since the famous work of S. Lie [39] it is well known that smooth point transformations with continuous parameter admitted by linear or nonlinear partial differential equations (PDEs) can be found algorithmically using Lie group analysis. The procedure that helps to find a symmetry group admitted by a PDE is well described in many textbooks, for example, we can recommend [46], [33] or [10] to the interested reader. Yet practical application of these procedures is connected with tedious and voluminous calculations which can be only slightly facilitated with the help of modern computer packages. For example, preparing this work we used the program **IntroToSymmetry** to obtain the determining system of equations. Solving these determining systems of partial differential equations is usually rather hard and can rarely be done by an algorithm, but the possibility to find the system of determining equations facilitates the work of a researcher since the systems are quite voluminous. For example, in the studied cases the systems had more then a hundred equations.

Once the Lie algebra admitted by the studied PDE is found one can find all non equivalent variable substitutions which reduce the dimension of the given PDE, if there are any. The found Lie algebra admitted by the PDE also generates the corresponding symmetry group of this equation. Using the corresponding exponential map of the adjoint representation of the considered Lie algebra we can find the symmetry group or corresponding subgroups of the equation as well. We do not have to look for an explicit form of the symmetry group to find reductions of the studied PDEs and invariant solutions of the equations. It is enough to know and to use the properties of the symmetry algebra which corresponds to the admitted symmetry group. The optimal system of subalgebras of this algebra gives rise to a complete set of non equivalent substitutions and as a result a set of different reductions of the studied PDE.

The solutions of reduced PDEs are called invariant solutions because they are invariant under the action of a given subgroup. One of the goals of this work is to find the admitted Lie algebras for PDEs describing value function and investment and consumption strategies for a portfolio with an illiquid asset that is sold in a random moment of time with a prescribed liquidation time distribution. We find the admitted Lie algebras for a general liquidation time distribution in cases of HARA and log utility functions and formulate corresponding theorems. We provide here the optimal system of subalgebras for a general case of a liquidation time distribution in both cases of HARA and logarithmic utility functions. We separately regard a case of an exponential distribution of a liquidation time where the corresponding PDE admits an extended Lie algebra. It leads to certain distinguishing properties that give rise to non trivial reductions of three dimensional PDEs to two dimensional equations and even to ordinary differential equations in some cases. We describe all non equivalent substitutions, provide the reductions and the corresponding lower dimensional equations as well as the corresponding allocation-consumption strategies.These lower dimensional equations can be used for further studies of portfolio optimization problems in similar way as one has done it before for the known substitutions in other cases.

Let us here briefly describe the method of Lie group analysis of PDEs. In the next Section we briefly introduce the ideas and notations needed for Lie group analysis of PDEs. The details can be found in many textbooks we prefer the notation used in [10].

5.1 Lie group analysis of PDEs. Notations and terminology

For this section we introduce some notation just to illustrate how Lie group analysis is applied to the PDEs. This is just a short overview and the interested reader can find a number of examples and a detailed theoretical background in several classical books, i.e. [27], [33], [46] and [39]. Let us note here that there are just a few types of PDEs that are frequently used in financial mathematics. The most typical equations for this area are second order parabolic PDEs. Here we cover an absolute minimum of the ideas and methods, that can facilitate our future research. Since we do not regard the problems that have more than three dimensions we recommend to keep it in mind as we formulate the definitions and the theorems for multi-dimensional cases, yet we try to cover the most general formulations anyway. Some definitions and statements that we find too voluminous we try to restrict to the simplified cases that are satisfy the purposes of this particular work. Later in this and other forthcoming chapters we apply introduced methods to the Equation (3.11) in cases of logarithmic and HARA utility and carry out complete Lie analysis of the problem.

Since this section is mostly informative and we talk here about the classical results, obtained without any connection with the problem of illiquidity a notation within this section is independent from the notation that we have used before or use after it. Let us talk about it in detail now.

We restrict ourself to the ideas that are needed to work with a single PDE but an interested reader can find in [46] that, in fact, the methods that we describe further can be applied to the systems of PDEs just with some little alterations, mostly a more sophisticated notation is needed.

Let us denote the independent variables as $(x_1, \ldots, x_p) \in X \cong \mathbb{R}^p$ and the dependent variable as $u \in V \cong \mathbb{R}$. The space $\mathcal{M} = X \times V$ (or an open subset $\mathcal{M} \subset X \times V$) would be called a base space. A notation of different partial derivatives is usually facilitated with the help of a so-called *multi-index* that can be defined as follows. Later we use the notations, definitions and Theorems' formulations from [10].

Definition 2. (Order of a multi-index, [10]) *Let $J = (j_1, \ldots, j_k)$ be an unordered k-tuple of integers with entries j_k, $1 \leq j_k \leq p$. The order of such a multi-index, which we denote as $\#J = |J| = k$, is defined as*

$$\#J = \#j_1 + \#j_2 + \ldots + \#j_k. \tag{5.1}$$

Let us also denote

$$\tilde{J} = (\tilde{j}_1, \ldots, \tilde{j}_p), \tag{5.2}$$

where $\tilde{j}_i = \#j_i$. Then we can introduce $\tilde{J}! = \tilde{j}_1! \cdot \ldots \cdot \tilde{j}_p!$.

Let us now consider the space that we denote as V_1 of all first derivatives of the variable $u(x) \in V$ with respect all dependent variables, $V_1 \subseteq \mathbb{R}^p$,

$$\left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_p}\right) \in V_1 \cong \mathbb{R}^p.$$

Analogously we can describe the space V_2 that would consist of all second order derivatives of the variable $u(x)$, more specifically

$$\left(\frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_p \partial x_p}\right) \in V_2 \cong \mathbb{R}^{N_2}, \quad N_2 = \binom{p+2-1}{2}.$$

We can act analogously up to the space V_k, that would be correspondingly a space of all k-th order derivatives of the variable $u(x)$, i.e.

$$\left(\frac{\partial^k u}{\partial {x_1}^k}, \ldots, \frac{\partial^k u}{\partial {x_p}^k}\right) \in V_k \cong \mathbb{R}^{N_k}, \quad N_k = \binom{p+k-1}{k}. \tag{5.3}$$

The space $V^{(n)}$ can be defined as follows

$$V^{(n)} = V \times V_1 \times \ldots \times V_n, \quad dim V^{(n)} = \binom{p+n}{n} = N^{(n)} \tag{5.4}$$

$V^{(n)}$ is the Cartesian product space. The coordinates in $V^{(n)}$ represent all the elements $u^{(n)}$, $n = 0, 1, \ldots$. The space $V^{(n)}$ is such that every element of it has $N^{(n)} = (1 + N_1 + N_2 + \ldots + N_n)$ components u_J. We use the multi-index $J = (j_1, \ldots, j_k)$ with $1 \leq j_k \leq p$, $\#J = k$ and $0 \leq k \leq n$ to denote the type of the corresponding partial derivative. For example, when $k = 0$, the value $u_{J=0}(x)$ coincides with the function $u(x)$ itself, i.e. $u_{J=0}(x) = u(x)$.

Definition 3. (Jet bundle, [10]) *The total space $X \times V^{(n)}$, denoted by $\mathcal{M}^{(n)}$, where coordinates represent the independent variables, the dependent variables and the derivatives of the dependent variables up to order n is called the n-th order jet bundle $j^{(n)}$ of the base space $\mathcal{M}$*

$$\mathcal{M}^{(n)} = \mathcal{M} \times V_1 \times \ldots \times V_n, \tag{5.5}$$

or the n-th prolongation of $\mathcal{M}$.

For a smooth, real-valued function $f(x) = f(x_1, x_2, \ldots, x_p)$ we denote a partial derivative of the order k as

$$\partial_J f(x) = \frac{\partial^k f(x)}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}}, \;\; J = (j_1, j_2, \ldots, j_k), \;\; \#J = k, \tag{5.6}$$

where $1 \leq j_i \leq p$, $i = 1, 2, \ldots, p$.

Definition 4. (n-th prolongation of f, [10]) *Given a smooth function $u = f(x)$, with $x \in X$ and $u \in V$, so $f : X \to V$. There exists an induced function*

$$u^{(n)} = \mathrm{pr}^{(n)} f(x)$$

called the n-th prolongation of $f(x)$, $x \in X \subset \mathbb{R}^p$ with values in $V^{(n)}$. Here we define the components $u^{(n)}$ by

$$u_J = \partial_J f(x), \tag{5.7}$$

i.e. $\mathrm{pr}^{(n)} f$ is a map from X to the space $V^{(n)}$.

To work with the PDEs of the n-th order we need to introduce some more theoretical concepts, for instance, a solution subvariety or a natural projection in n-order jet bundle. These definitions can also be found in the books such as [46] or [47], but here we cover them briefly, just in order to communicate general ideas needed for our further research. In a form that corresponds to our purposes these definitions were provided in [10], so we use these definitions here verbatim.

Definition 5. (Partial differential equation of the n-th order, [10]) *We denote the n-th order PDE on dependent variable $u(x)$ and p independent variables as*

$$\Delta(x, u^{(n)}) = 0, \tag{5.8}$$

involving $x = (x_1, \ldots, x_p)$, $u(x)$ *and derivatives of* $u(x)$ *with respect to* x_i, $i = 1, 2, \ldots, p$ *up to the order* n. *We consider* Δ *as a smooth map from the jet bundle* $\mathcal{M}^{(n)} = X \times V^{(n)}$ *to some Euclidean space*

$$\Delta : \mathcal{M}^{(n)} = X \times V^{(n)} \to \mathbb{R}.$$

The equality $\Delta = 0$ *determines a subvariety*

$$\mathcal{S}_\Delta = \{(x, u^{(n)}) : \Delta(x, u^{(n)}) = 0\} \subset \mathcal{M}^{(n)} = X \times V^{(n)}$$

of the total jet bundle. The differential Equation (5.8) is identified with its subvariety $\mathcal{S}_\Delta$. *The subvariety is called the solution subvariety of the given differential equation.*

Definition 6. (Solution, [10]) *A smooth solution of the given PDE is a smooth function* $u = f(x)$, *such that*

$$\Delta(x, \mathrm{pr}^{(n)} f(x)) = 0, \tag{5.9}$$

whenever x *lies in the domain of* $f(x)$.

Let us note here that a solution is to be a smooth function $u = f(x)$ in order to satisfy the condition on the graph of the n-th prolongation of f that should lie in the subvariety $\mathcal{S}_\Delta$. If $u = f(x)$ is actually a smooth function defined at the point $x_0 \in \mathcal{M}$ then it has a graph defined in $\mathcal{M}^{(n)}$ as

$$u^{(n)}(x_0) = \mathrm{pr}^{(n)} f(x_0),$$

with the components that are correspondingly determined as $u_J(x_0) = \partial_J f(x)|_{x=x_0}$, $\#J = k,\ \ 1 \le k \le n$. Around the point $x_0 \in X$ we can look at the n-th order Taylor polynomial that looks as

$$f(x) = \sum_J \frac{1}{\tilde{J}!} u_J(x_0) \cdot (x - x_0)^J, \quad \#J = k, \ \ 1 \le k \le n,$$

where the notation goes as follows $(x - x_0)^J = (x_{j_1} - x_{0,j_1})(x_{j_2} - x_{0,j_2}) \cdots (x_{j_k} - x_{0,j_k})$ with $J = (j_1, j_2, \ldots, j_k)$ and $x_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,p})$.

Let g be an element of a transformation group $g \in G$. Assuming that the group transformation $g \circ f$ is well-defined in the neighborhood of x_0 we obtain the following

$$(\tilde{x}_0, \tilde{u}_0) = g \circ (x_0, u_0), \quad u_0 = f(x_0).$$

The action of the prolonged group transformation could be defined as $\mathrm{pr}^{(n)}g$ at the point $(x_0, {u_0}^{(n)})$. After the evaluation of the derivatives of the transformed function $g \circ f$ in $\tilde{x}_0$ we obtain

$$\mathrm{pr}^{(n)}g \circ (x_0, {u_0}^{(n)}) = (\tilde{x}_0, {\tilde{u}_0}^{(n)}),$$

where ${\tilde{u}_0}^{(n)} = (\mathrm{pr}^{(n)}g \circ f)(\tilde{x}_0)$.

Definition 7. (Natural projection, [46]) *We call the map*

$$\Pi^n_k : \mathcal{M}^{(n)} \to \mathcal{M}^{(k)} \tag{5.10}$$

with

$$\Pi^n_k(x, u^{(n)}) = (x, u^{(k)}), \tag{5.11}$$

where $u^{(k)}$ only consists of the components $u_{\alpha,J}$, $\#J \leq k,\ k < n$, of $u^{(n)}$ itself, a natural projection.

Remark 6. *A prolongation of the 0-order is by definition $\mathrm{pr}^{(0)}G = G$. If we look in the derivatives up to order $k \leq n$ only, i.e. $(x, u^{(k)})$, the action of $\mathrm{pr}^{(k)}$ coincides with the prolongation $\mathrm{pr}^{(k)}G$ that was described above and we get the following relation*

$$\Pi^n_k \circ \mathrm{pr}^{(n)}g = \mathrm{pr}^{(k)}g, \quad \forall g \in G.$$

Theorem 8. *(Invariance of differential equations, [46])*
Let $\mathcal{M}$ be an open subset of $X \times V$ and suppose $\Delta(x, u^{(n)}) = 0$ is an n-th order PDE defined over $\mathcal{M}$ with corresponding subvariety $\mathcal{S}_\Delta \subset \mathcal{M}^{(n)}$. Suppose G is a local continuous group of point transformations acting on $\mathcal{M}$, which prolongation leaves $\mathcal{S}_\Delta$ invariant, which means that whenever $(x, u^{(n)}) \in \mathcal{S}_\Delta$, we have $\mathrm{pr}^{(n)}g \circ (x, u^{(n)}) \in \mathcal{S}_\Delta$ for all $g \in G$, such that this expression is defined.
Then G is a symmetry group of the given PDE.

Proof: Suppose $u = f(x)$ is a local solution to $\Delta(x, u^{(n)}) = 0$. Then the graph of this function which is denoted as $\Gamma^{(n)}_f = \{x, \mathrm{pr}^{(n)}f(x)\}$ lies entirely within $\mathcal{S}_\Delta$.

If $g \in G$ and $g \circ f$ is well-defined, then $\Gamma^{(n)}_{g \circ f} = \mathrm{pr}^{(n)}g(\Gamma^{(n)}_f)$. Because $\mathcal{S}_\Delta$ is invariant under $\mathrm{pr}^{(n)}g$, we have again that $\mathrm{pr}^{(n)}(g \circ f)$ lies entirely in $\mathcal{S}_\Delta$ and is a solution to the system $\Delta(x, u^{(n)}) = 0$ as well. □

Definition 8. (Prolongation of vector fields, [10]) *Let $\mathcal{M} \subset X \times V$ be an open subspace and suppose $\mathbf{U}$ is a vector field (an infinitesimal generator of a transformation group) on $\mathcal{M}$ with a corresponding (local) one-parameter group $G = \exp(\varepsilon \mathbf{U})$. The n-th prolongation of $\mathbf{U}$, denoted by $\mathrm{pr}^{(n)}\mathbf{U}$ or $\mathbf{U}^{(n)}$, will be a vector field on the jet bundle $\mathcal{M}^{(n)}$, and it is defined to be the infinitesimal generator of the corresponding prolonged group $\mathrm{pr}^{(n)}G = G^{(n)} = \mathrm{pr}^{(n)}\left(\exp(\varepsilon \mathbf{U})\right)$*

$$\mathrm{pr}^{(n)}\mathbf{U}\Big|_{(x,u^{(n)})} = \frac{d}{d\varepsilon}\left(\mathrm{pr}^{(n)}\exp(\varepsilon\mathbf{U})(x,u^{(n)})\right)\Bigg|_{\varepsilon=0} \tag{5.12}$$

for any $(x, u^{(n)}) \in \mathcal{M}^{(n)}$.

Remark 7. *An infinitesimal generator also known as a prolonged vector field has the following structure*

$$\mathbf{U}^{(n)} = \mathrm{pr}^{(n)}\mathbf{U} = \sum_{i=1}^{p} \xi_i \frac{\partial}{\partial x_i} + \sum_{\substack{J \\ 0 \le \#J \le n}} \Phi^J \frac{\partial}{\partial u_J}, \tag{5.13}$$

where $x = (x_1, \dots, x_p)$ are independent variables and $u = u(x)$ is the dependent one. The coefficients ξ_i, $i = 1, 2, \dots, p$ depend on variables $x_i, i = 1, 2, \dots, p$ and u, the function Φ^J depends on variables $x_i, i = 1, 2, \dots, p$ and derivatives of u up to the order $\#J$.

Technically, the components ξ_i, Φ^J could depend on all variables of this jet bundle, if we work with an arbitrary linear differential operator (vector field). However, if $\mathbf{V}$ is a n-th order prolongation of $\mathbf{U}$, i.e. $\mathbf{V} = \mathrm{pr}^{(n)}\mathbf{U}$, all coefficients of this vector field are actually determined by the coefficients of the original vector field $\mathbf{U}$, that is defined on the base space $\mathcal{M}$. Since $\mathcal{M}^{(0)} = \mathcal{M}$ and $\mathrm{pr}^{(0)}\mathbf{U} = \mathbf{U}$ we get

$$\xi_i = \xi_i(x_1, \dots, x_p, u), \quad \Phi^0 = \Phi(x_1, \dots, x_p, u).$$

Taking into consideration

$$\Pi_k^n \circ \mathrm{pr}^{(n)} g = \mathrm{pr}^{(k)} g, \qquad \forall g \in G,$$

each coefficient of the vector field $\mathrm{pr}^{(k)}\mathbf{U}$ can depend on k-th derivatives of u or other derivatives of a lower order. For instance, if $\#J = k$ we have $\Phi^J = \Phi^J(x, u^{(k)})$.

Theorem 9. *(Determining equations, [10])*
Suppose $\Delta(x, u^{(n)}) = 0$, is a PDE defined on $\mathcal{M} \subset X \times V$. If G is a local continuous group of point transformations acting on $\mathcal{M}$ generated by Lie algebra L and

$$\mathrm{pr}^{(n)}\mathbf{U}\left(\Delta(x, u^{(n)})\right) = 0, \quad \forall \mathbf{U} \in L,$$

whenever $\Delta(x, u^{(n)}) = 0$, then G is a symmetry group of the system.

Definition 9. (Total differentiation, [10]) *Let us introduce a total differentiation given by the following formal infinite sums*

$$\begin{aligned} D_i &= \frac{\partial}{\partial x_i} + \sum_J u_{i,J} \frac{\partial}{\partial u_J} \\ &= \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + \sum_{i_1=1}^{p} u_{ii_1} \frac{\partial}{\partial u_{i_1}} + \dots \\ &\quad + \sum_{i_1=1}^{p} \sum_{i_2=1}^{p} \dots \sum_{i_r=1}^{p} u_{ii_1i_2\dots i_r} \frac{\partial}{\partial u_{i_1i_2\dots i_r}} \cdots, \quad i = 1, \dots, p, \quad 0 \le \#J \le \infty. \end{aligned}$$

The variables x_i are called independent variables and $u(x) = u(x_1, x_2, \dots, x_p)$ is a dependent variable with derivatives denoted as $u_{i,J}$.

The rules of general differentiation are also applicable to total derivatives. Let us list the most important of them here according to [10].

Theorem 10. *(Faá di Bruno's formula, chain rule for a composite function, [10])*
Let $p = 1$. If $u = f(y)$ and $y = \varphi(x)$, the k-th order derivative of the function $f(y)$ is given, in terms of $y', \dots, y^{(k)}$ and $f' = \frac{df}{dy}, \dots, f^{(k)} = \frac{d^k f}{dy^k}$, by the formula

$$D_x^k(f) = \sum_{\substack{l_1,\dots,l_k \ge 0 \\ l_1+2l_2+\dots+kl_k=k \\ p=l_1+\dots+l_k \ge 0}} \frac{k!}{l_1! \dots l_k!} f^{(p)} \left(\frac{y'}{1!}\right)^{l_1} \left(\frac{y''}{2!}\right)^{l_2} \dots \left(\frac{y^{(k)}}{k!}\right)^{l_k}.$$

Definition 10. (Total derivative, [46]) *Let $\Phi(x, u^{(n)})$ ($\dim X = p, \dim V = 1$) be a smooth function of $x \in X$, $u \in V$ and derivatives of u up to the order n, defined on an open subset $\mathcal{M}^{(n)} = X \times V^{(n)}$. The total derivative of Φ with respect to x_i is a unique smooth function $D_i\Phi(x, u^{(n+1)})$, defined on $\mathcal{M}^{(n+1)}$, depending on* *derivatives of u up to the order $(n+1)$. It has the property that, if $u = f(x)$ is an arbitrary smooth function*

$$D_i\Phi(x, \mathrm{pr}^{(n+1)}f(x)) = \frac{\partial}{\partial x_i}\left(\Phi(x, \mathrm{pr}^{(n)}f(x))\right), \tag{5.14}$$

where $D_i\Phi$ is obtained by differentiating Φ with respect to x_i while treating $u(x)$ and its derivatives as functions of x,

$$D_i\Phi = \frac{\partial\Phi}{\partial x_i} + \sum_{\substack{J \\ 0\leq \#J\leq n}} u_{i,J}\frac{\partial\Phi}{\partial u_J}, \tag{5.15}$$

for $J = (j_1, \ldots, j_k)$, $u_{i,J} = \frac{\partial u_J}{\partial x_i} = \frac{\partial^{k+1}u}{\partial x_i \partial x^{j_1} \partial x^{j_k}}$.

Theorem 11. *[10] Let*

$$\mathbf{V} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x_i} + \Phi(x,u)\frac{\partial}{\partial u}$$

be a vector field (an infinitesimal generator), defined on an open subset $\mathcal{M} \subset X \times V$. The n-th prolongation of $\mathbf{V}$ is the vector field

$$\mathrm{pr}^{(n)}\mathbf{V} = \mathbf{V} + \sum_{\substack{J \\ 0\leq\#J\leq n \\ 1\leq j_k\leq p}} \Phi^J(x, u^{(n)})\frac{\partial}{\partial u_J}, \tag{5.16}$$

defined on the jet bundle $\mathcal{M}^{(n)} \subset X \times V^{(n)}$. The coefficient functions Φ^J of $\mathrm{pr}^{(n)}\mathbf{V}$ are given by

$$\Phi^J(x, u^{(n)}) = D_J\left(\Phi - \sum_{i=1}^{p}\xi^i u_i\right) + \sum_{i=1}^{p}\xi^i u_{i,J}, \tag{5.17}$$

where $u_i = \frac{\partial u}{\partial x_i}$, $u_{i,J} = \frac{\partial u_J}{\partial x_i}$.

These main techniques and definitions we apply to the HJB equations that correspond to the optimization problems with different utility functions and different liquidation time distributions. In the next Section we come back to the standard notation that is used throughout this work and use the ideas introduced here later.

5.2 Lie group analyses of the problem with a general liquidation time distribution and different utility functions

After a formal maximization of (3.12) and (3.13) for the chosen utility function the equation (3.11) becomes a three dimensional non-linear PDE. As we have already said in this particular work we regard two different utility functions and now we look at the cases of HARA utility and log utility separately.

5.2.1 The case of HARA utility function

It is well known that a utility function $U(c)$ where the risk tolerance $R(c)$ is defined as $R(c) = -\frac{U'(c)}{U''(c)}$ and is a linear function of c, is called a HARA (hyperbolic absolute risk aversion) utility function. In this work we use two types of utility functions: a HARA utility function $U^{HARA}(c)$ and the log-utility function $U^{LOG}(c) = \log(c)$. Let us note here that the log-utility function is sometimes described as a limit case of HARA utility function, see, for instance, [13] or [49]. One can indeed choose HARA utility in such a way that allows a formal transition from HARA utility to log-utility as parameter γ of HARA utility goes to zero, but in general this transition does not hold for any form of HARA utility. We will demonstrate this transition on different levels and because of that further we work with HARA utility in the form

$$U^{HARA}(c) = \frac{1-\gamma}{\gamma}\left(\left(\frac{c}{1-\gamma}\right)^{\gamma} - 1\right), \tag{5.18}$$

with the risk tolerance $R(c) = \frac{c}{1-\gamma}$, $0 < \gamma < 1$. One can easily see that as $\gamma \to 0$ HARA-utility function written as (5.18) tends the to log-utility

$$U^{HARA} \underset{\gamma \to 0}{\longrightarrow} U^{LOG}. \tag{5.19}$$

The HJB equation (3.11) where we insert the HARA utility in the form (5.18) after formal maximization procedure looks as follows

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ - & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ + & \frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma}\overline{\Phi}(t) = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{5.20}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function V

$$\pi(t,l,h) = -\frac{\eta\rho\sigma h V_{lh}(t,l,h) + (\alpha-r)V_l(t,l,h)}{\sigma^2 V_{ll}(t,l,h)}, \tag{5.21}$$

$$c(t,l,h) = (1-\gamma)V_l(t,l,h)^{-\frac{1}{1-\gamma}}\overline{\Phi}(t)^{\frac{1}{1-\gamma}}. \tag{5.22}$$

Equation (5.20) is a nonlinear three dimensional PDE with three independent variables t, l, h. To reduce the dimension of the equation (5.20) we use Lie group analysis, that allows us to find the generators of the corresponding symmetry algebra admitted by this equation. We have already described the methods that we use here in Section 5.1. If the reader is interested in a more detailed description or in some practical examples we address him to PDEs in [10]. Here we formulate the main theorem of Lie group analysis for the optimization problem with HARA type utility function.

Theorem 12. *The equation (5.20) admits the three dimensional Lie algebra L_3^{HARA} spanned by generators $L_3^{HARA} = <\mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3>$, where*

$$\begin{aligned} &\mathbf{U}_1 = \frac{\partial}{\partial V}, \qquad \mathbf{U}_2 = e^{rt}\frac{\partial}{\partial l}, \\ &\mathbf{U}_3 = l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)\frac{\partial}{\partial V}, \end{aligned} \tag{5.23}$$

for any liquidation time distribution. Moreover, if and only if the liquidation time distribution has the exponential form, i.e. $\overline{\Phi}(t) = de^{-\kappa t}$, where d, κ are constants the studied equation admits a four dimensional Lie algebra L_4^{HARA} with an additional generator

$$\mathbf{U}_4 = \frac{\partial}{\partial t} - \kappa V\frac{\partial}{\partial V}, \tag{5.24}$$

i.e. $L_4^{HARA} =< \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4 >$.
Except finite dimensional Lie algebras (5.23) and (5.24) correspondingly equation (5.20) admits also an infinite dimensional algebra $L_\infty =< \psi(h,t)\frac{\partial}{\partial V} >$ *where the function* $\psi(h,t)$ *is any solution of the linear PDE*

$$\psi_t(h,t) + \frac{1}{2}\eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta)h\psi_h(h,t) = 0. \tag{5.25}$$

The Lie algebra L_3^{HARA} *has the following non-zero commutator relations*

$$[\mathbf{U}_1, \mathbf{U}_3] = \gamma \mathbf{U}_1, \qquad [\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_2 \tag{5.26}$$

The Lie algebra L_4^{HARA} *has the following non-zero commutator relations*

$$[\mathbf{U}_1, \mathbf{U}_3] = \gamma \mathbf{U}_1,\ [\mathbf{U}_1, \mathbf{U}_4] = -\kappa \mathbf{U}_1,\ [\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_2,\ [\mathbf{U}_2, \mathbf{U}_4] = -r\mathbf{U}_2 \tag{5.27}$$

Remark 8. *The found Lie algebra describes the symmetry property of the equation (5.20) for any function* $\overline{\Phi}(t)$. *In Chapter 3 and Chapter 4 we have proved the theorem for existence and uniqueness of the solution of HJB equation for a liquidation time distribution which* $\overline{\Phi}(t) \sim e^{-\kappa t}$ *or faster as* $t \to \infty$, *therefore we will regard this type of the distribution studying the analytical properties of the equation further on.*

Proof 1. *Using Definition 3 in the previous Section we introduce the second order jet bundle* $j^{(2)}$ *and present the equation (5.20) in the form*

$$\Delta(l, h, t, V, V_l, V_h, V_t, V_{ll}, V_{ll}, V_{lh}, V_{hh}) = 0$$

as a function of these variables in the jet bundle $j^{(2)}$. *We look for generators of the admitted Lie algebra in the form*

$$\mathbf{U} = \xi_1(l,h,t,V)\frac{\partial}{\partial l} + \xi_2(l,h,t,V)\frac{\partial}{\partial h} + \xi_3(l,h,t,V)\frac{\partial}{\partial t} + \eta_1(l,h,t,V)\frac{\partial}{\partial V}, \tag{5.28}$$

where the functions $\xi_1, \xi_2, \xi_3, \eta_1$ *can be found using the over determined system of determining equations*

$$\mathbf{U}^{(2)}\Delta(l, h, t, V, V_l, V_h, V_t, V_{ll}, V_{ll}, V_{lh}, V_{hh})|_{\Delta=0} = 0, \tag{5.29}$$

where $\mathbf{U}^{(2)}$ *is the second prolongation of* $\mathbf{U}$ *in* $j^{(2)}$. *We look at the action of* $\mathbf{U}^{(2)}$ *on* $\Delta(l, h, t, V, V_l, V_h, V_t, V_{ll}, V_{ll}, V_{lh}, V_{hh})$ *on its solution subvariety* $\Delta = 0$

and obtain an overdetermined system of PDEs on the functions ξ_1, ξ_2, ξ_3 *and* η_1 *from (5.28). This system has 137 PDEs on the functions* $\xi_1, \xi_2, \xi_3, \eta_1$. *The most of them are trivial and lead to following conditions on the functions*

$$\begin{aligned}(\xi_1)_l &= a_1(t),\ (\xi_1)_h = 0,\ (\xi_1)_V = 0,\\ (\xi_2)_l &= 0,\ (\xi_2)_V = 0,\\ (\xi_3)_l &= 0,\ (\xi_3)_h = 0,\ (\xi_3)_V = 0,\\ (\eta_1)_l &= 0,\ (\eta_1)_V = d_1(h,t).\end{aligned}$$

This basically means that the unknown functions have the following structure

$$\begin{aligned}\xi_1(l,h,t,V) &= a_1(t)l + a_2(t),\quad \xi_2(l,h,t,V) = \xi_2(h,t),\quad \xi_3(l,h,t,V) = \xi_3(t),\\ \eta_1(l,h,t,V) &= d_1(h,t)V + d_2(h,t). \qquad (5.30)\end{aligned}$$

Here $a_1(t)$ *and* $d_1(h,t)$ *are some functions which will be defined later. To find the unknown functions* $a_1(t), a_2(t), \xi_2(h,t), d_1(h,t), d_2(h,t)$ *we should have a closer look on the non-trivial equations of the obtained system, that are left. After all simplifications we get the system of seven PDEs*

$$\begin{aligned}2(\xi_2 - h\xi_{2h}) + h\xi_{3t} &= 0, \qquad (5.31)\\ (1-\gamma)\overline{\Phi}\left(\eta_{1V} - \xi_3 - \xi_{3t}\frac{\overline{\Phi}_t}{\overline{\Phi}}\right) + \gamma \boldsymbol{L}(\eta_1) &= 0, \qquad (5.32)\\ \eta_{1V} - \gamma\xi_{1l} - \frac{\overline{\Phi}_t}{\overline{\Phi}}\xi_3 - (1-\gamma)\xi_{3t} &= 0, \qquad (5.33)\\ (\alpha - r)\xi_{3t} + 2\eta\rho h\eta_{1hV} &= 0,\\ (\alpha - r)(\xi_2 - h\xi_{2h} + h\xi_{3t}) + \eta\rho\sigma^2h^2\eta_{1hV} &= 0,\\ r\xi_1 - \xi_{1t} - \xi_{1l}(\delta h + rl) + \delta\xi_2 + (\delta h + rl)\xi_{3t} &= 0,\\ (\mu - \delta)(\xi_2 - h\xi_{2h} + h\xi_{3t}) - \xi_{2t} - \frac{1}{2}\eta^2h^2\eta_{1hh} &= 0,\end{aligned}$$

where $\boldsymbol{L} = \frac{\partial}{\partial t} + \frac{1}{2}\eta^2h^2\frac{\partial^2}{\partial h^2} - (\delta - \mu)h\frac{\partial}{\partial h}$ *and* $\xi_1, \xi_2, \xi_3 = const$ *and* η_1 *satisfy (5.30). Using (5.30) we obtain a simplified system.*
Solving the system for an arbitrary function $\overline{\Phi}(t)$ *we obtain*

$$\begin{aligned}\xi_1 &= b_1l + a_2e^{rt}, \qquad (5.34)\\ \xi_2 &= b_1h,\\ \xi_3 &= 0,\\ \eta_1 &= b_1\gamma V + d_2 - b_1(1-\gamma)\int\overline{\Phi}(t)dt + d_1(h,t).\end{aligned}$$

The equations (5.34) contain three arbitrary constants a_2, b_1, d_2 and a function $d_1(h,t)$ which is an arbitrary solution of $\boldsymbol{L}d_1(h,t) = 0$. Formulas (5.34) define three generators of finite dimensional Lie algebra L_3^{HARA} (5.23) and an infinitely dimensional algebra L_∞ as it was described in Theorem 12.

If we assume that in the equations (5.32) and (5.33) the expression $\frac{\overline{\Phi}_t}{\overline{\Phi}} = const$, i.e. the liquidation time is exponentially distributed we additionally obtain the fourth symmetry (5.24). It is a unique case when Lie algebra L_3^{HARA} has any extensions. •

5.2.2 The case of the log-utility function

A logarithmic utility function could be regarded as a limit case of HARA-utility (5.19). Yet certain properties of the logarithm make this particular case rather popular therefore we analyze it separately.

The whole approach is very similar to the method described in Section 5.2.1 therefore we omit some details here. In the case of the log-utility function the HJB equation after the formal maximization procedure will take the following form

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ - & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ - & \overline{\Phi}(t)\left(\log V_l - \log\overline{\Phi}(t) + 1\right) = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{5.35}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function V

$$\pi(t,l,h) = -\frac{\eta\rho\sigma h V_{lh}(t,l,h) + (\alpha-r)V_l(t,l,h)}{\sigma^2 V_{ll}(t,l,h)}, \tag{5.36}$$

$$c(t,l,h) = \frac{\overline{\Phi}(t)}{V_l(t,l,h)}. \tag{5.37}$$

Remark 9. *We choose the form of HARA-utility in such a way that (5.19) holds. Now we see that the maximization procedure that transforms HJB equation to PDE preserves this property as well. If we formally take a limit of (5.20) as $\gamma \to 0$ we obtain (5.35).*

As it turns out analogously to the previous chapter one can formulate the main theorem of Lie group analysis for this PDE.

Theorem 13. *The equation (5.35) admits the three dimensional Lie algebra L_3^{LOG} spanned by generators $L_3^{LOG} =< \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3 >$, where*

$$\mathbf{U}_1 = \frac{\partial}{\partial V}, \qquad \mathbf{U}_2 = e^{rt}\frac{\partial}{\partial l},$$
$$\mathbf{U}_3 = l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} - \int \overline{\Phi}(t)dt\frac{\partial}{\partial V}, \tag{5.38}$$

for any liquidation time distribution. Moreover, if and only if the liquidation time distribution has the exponential form, i.e. $\overline{\Phi}(t) = de^{-\kappa t}$, where d, κ are constants, the studied equation admits a four dimensional Lie algebra L_4^{LOG} with an additional generator

$$\mathbf{U}_4 = \frac{\partial}{\partial t} - \kappa V\frac{\partial}{\partial V}, \tag{5.39}$$

i.e. $L_4^{LOG} =< \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4 >$.
Except finite dimensional Lie algebras L_3^{LOG} and L_4^{LOG} correspondingly the equation (5.35) admits also an infinite dimensional algebra $L_\infty =< \psi(h,t)\frac{\partial}{\partial V} >$ where the function $\psi(h,t)$ is any solution of the linear PDE

$$\psi_t(h,t) + \frac{1}{2}\eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta)h\psi_h(h,t) = 0. \tag{5.40}$$

The Lie algebra L_3^{LOG} has one non-zero commutator relation $[\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_2$.
The Lie algebra L_4^{LOG} has the following non-zero commutator relations

$$[\mathbf{U}_1, \mathbf{U}_4] = -\kappa\mathbf{U}_1, \quad [\mathbf{U}_2, \mathbf{U}_3] = \mathbf{U}_2, \quad [\mathbf{U}_2, \mathbf{U}_4] = -r\mathbf{U}_2.$$

Remark 10. *If we compare the form of Lie algebras generators in the cases of HARA and log utilities, i.e. formulas (5.23) and (5.38) as well as (5.24) and (5.39), we can see that the formal limit procedure holds for them as well and the generators for HARA-utility transfer to generators for log-utility under a formal limit $\gamma \to 0$.*

Proof 2. *Acting analogously to the proof of the Theorem 13 we look for the generators of the admitted Lie algebra in the form (5.28). A corresponding determining system obtained analogously to (5.29) has 130 equations on the functions ξ_1, ξ_2, ξ_3*

and η_1. The most of these equation are trivial and we can easily solve them. This way we obtain

$$\begin{aligned}
(\xi_1)_l &= b_1,\ (\xi_1)_h = 0,\ (\xi_1)_V = 0, \\
(\xi_2)_l &= 0,\ (\xi_2)_V = 0, \\
(\xi_3)_l &= 0,\ (\xi_3)_h = 0,\ (\xi_3)_V = 0, \\
(\eta_1)_l &= 0,\ (\eta_1)_V = d_1(h,t),
\end{aligned} \tag{5.41}$$

where b_1 is a constant and $d_1(h,t)$ is a function to be determined. The remaining equations can be rewritten as

$$\begin{aligned}
r\xi_1 - (rl+\delta h)\xi_{1h} - \xi_{1t} + \delta\xi_2 + (rl+\delta h)\xi_{3t} &= 0, \\
(\mu-\delta)(\xi_2 - h\xi_{2h} + h\xi_{3t}) - \xi_{2t} - \frac{1}{2}\eta^2 h^2 \xi_{2hh} + \eta^2 h^2 \eta_{1hV} &= 0, \\
\xi_2 - h\xi_{2h} + \frac{1}{2}h\xi_{3t} &= 0, \\
\xi_{3t} + \frac{\overline{\Phi}_t}{\overline{\Phi}}\xi_3 - \eta_{1V} &= 0, \\
\overline{\Phi}\xi_{1l}\overline{\Phi}_t \log\overline{\Phi} + \overline{\Phi}(\log\overline{\Phi} - 1)\xi_{3t} - \overline{\Phi}\log\overline{\Phi}\eta_{1V} + \boldsymbol{L}(\eta_1) &= 0, \\
(\alpha - r)\xi_{3t} + 2\eta\rho h\eta_{1hV} &= 0, \\
(\alpha - r)(\xi_2 - h\xi_{2h} + h\xi_{3t}) + \eta\rho\sigma^2 h^2 \eta_{1hV} &= 0,
\end{aligned} \tag{5.42}$$

where $\boldsymbol{L} = \frac{\partial}{\partial t} + \frac{1}{2}\eta^2 h^2 \frac{\partial^2}{\partial h^2} - (\delta - \mu)h\frac{\partial}{\partial h}$ and $\xi_1 = \xi_1(l,t)$, $\xi_2 = \xi_2(h,t)$, $\xi_3 = \xi_3(t)$ and $\eta_1 = \eta_1(h,t,V)$ are described in (5.41). Inserting these functions ξ_1, ξ_2, ξ_3 and η_1 into (5.42) we obtain a simplified system of determining equations.

Solving the system for an arbitrary $\overline{\Phi}(t)$ we obtain the following solution

$$\begin{aligned}
\xi_1 &= b_1 l + a_2 e^{rt}, \\
\xi_2 &= b_1 h, \\
\xi_3 &= 0, \\
\eta_1 &= -b_1 \int \overline{\Phi}(t)dt + d_2 + d_1(h,t),
\end{aligned} \tag{5.43}$$

where $d_1(h,t)$ is an arbitrary solution of $\boldsymbol{L}d_1(h,t) = 0$ and a_2, b_1 and d_2 are arbitrary constants. This solution defines three different operators, that are listed in (5.38).

If and only if the expression $\frac{\overline{\Phi}'(t)}{\overline{\Phi}(t)}$ is a constant further denoted as κ the solution of the overdetermined system (5.42) is as follows

$$\begin{aligned}
\xi_1 &= b_1 l + a_2 e^{rt}, \\
\xi_2 &= b_1 h, \\
\xi_3 &= c_1, \\
\eta_1 &= -b_1 \int \overline{\Phi}(t)dt + d_2 + d_1(h,t) - c_1 \kappa V,
\end{aligned} \tag{5.44}$$

It means that just for the special exponential form of $\overline{\Phi}(t)$ we obtain an extension of the Lie algebra L_3^{LOG} to L_4^{LOG} with an additional generator $\mathbf{U}_4$ (5.39). In this way we proved the Theorem 13 and found the generators of L_3^{LOG} and L_4^{LOG} as given in (5.38) and, correspondingly in (5.39). •

5.3 Correspondence between models with HARA and logarithmic utility functions

The fact that logarithmic utility function can be regarded as a limit case of HARA utility is mentioned in several publications, for example, [54] or [49]. We have studied the issue in detail, yet we feel that it didn't get sufficient attention before, so in this section we briefly discuss the connection between HARA and logarithmic utilities in the context of portfolio optimization for a portfolio with an illiquid asset.

We have already written that if HARA utility has the form (5.18), it is clear that if $\gamma \to 0$ then indeed $U^{HARA} \underset{\gamma\to 0}{\longrightarrow} U^{LOG}$. This correspondence follows directly from the chosen form of HARA utility

$$U^{HARA}(c) = \frac{1-\gamma}{\gamma}\left(\left(\frac{c}{1-\gamma}\right)^{\gamma} - 1\right) \tag{5.45}$$

This also means that before a formal maximization the problem for HARA utility written in the form (3.11) tends formally to the problem formulated with logarithmic utility as $\gamma \to 0$. However, the fact that this correspondence is preserved after a procedure of formal maximization is not so obvious.

As we have written before the HJB equation (3.11) where we insert the HARA utility in the form (5.18) after a formal maximization procedure will take the form

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ - & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ + & \frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma}\overline{\Phi}(t) = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0, \end{aligned} \tag{5.46}$$

whereas the HJB equation (3.11) with logarithmic utility would be

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ - & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ - & \overline{\Phi}(t)\left(\log V_l - \log\overline{\Phi}(t) + 1\right) = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{5.47}$$

Let us check if the term

$$\frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma}\overline{\Phi}(t)$$

formally tends to $-\overline{\Phi}(t)\left(\log V_l - \log\overline{\Phi}(t)+1\right)$ as $\gamma \to 0$.

Indeed

$$\begin{aligned} & \lim_{\gamma\to 0}\frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma}\overline{\Phi}(t) \\ = & -\overline{\Phi}(t)\lim_{\gamma\to 0}(1-\gamma)\left(\frac{-(1-\gamma)\left(\frac{\overline{\Phi}(t)}{V_l(t,l,h)}\right)^{\frac{\gamma}{1-\gamma}} + 1}{\gamma}\right), \end{aligned}$$

and applying l'Hopital rule to the term in the brackets we obtain

$$\begin{aligned} & -\overline{\Phi}(t)\lim_{\gamma\to 0}(1-\gamma)\left(\left(\frac{\overline{\Phi}(t)}{V_l(t,l,h)}\right)^{\frac{\gamma}{1-\gamma}} - (1-\gamma)\left(\frac{\overline{\Phi}(t)}{V_l(t,l,h)}\right)^{\frac{\gamma}{1-\gamma}}\log\frac{\overline{\Phi}(t)}{V_l(t,l,h)}\right) \\ = & -\overline{\Phi}(t)\left(\log V_l - \log\overline{\Phi}(t)+1\right). \end{aligned}$$

This means that the correspondence between HARA utility and logarithmic utility function is preserved even after a procedure of formal maximization. We can show this correspondence even further and see it on the level of algebras admitted by the equations (5.46) and (5.47).

Let us briefly remind the reader that L_3^{HARA} (5.23) is spanned by the following operators

$$\mathbf{U}_1 = \frac{\partial}{\partial V}, \qquad \mathbf{U}_2 = e^{rt}\frac{\partial}{\partial l},$$
$$\mathbf{U}_3 = l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)\frac{\partial}{\partial V},$$

whereas L_3^{LOG} (5.38) is spanned by

$$\mathbf{U}_1 = \frac{\partial}{\partial V}, \qquad \mathbf{U}_2 = e^{rt}\frac{\partial}{\partial l},$$
$$\mathbf{U}_3 = l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} - \int \overline{\Phi}(t)dt\frac{\partial}{\partial V}.$$

Indeed, as $\gamma \to 0$ we can see that $U_3^{HARA} \underset{\gamma \to 0}{\longrightarrow} U_3^{LOG}$ and all other generators are the same in both cases (as well as $U_4^{HARA} \underset{\gamma \to 0}{\longrightarrow} U_4^{LOG}$ in the case of four dimensional algebras).

This means that for HARA utility defined as (5.18) when $\gamma \to 0$ there is a formal limit and this formal procedure does not only connect HARA and logarithmic utility functions but also HJB equations that correspond to these functions and Lie algebras admitted by these equations.

Before we go further let us make one more remark. In the next chapters we look for reductions of (5.20) and (5.35) using the admitted Lie algebras. Standard methods that are used to obtain corresponding invariants give us results in such a form that the correspondence between the problem with HARA and logarithmic utility, that we discussed above, may be destroyed after using certain reductions. Technically, it might be possible to choose invariants in such a form that this relation could be preserved even after a reduction, but we do.

Let us give here a brief example (we explain how the reductions are obtained in Chapter 6 and Chapter 7 correspondingly), here we just need the form of the reduced equations and the form of invariants, used for substitution.

Let us look on subalgebra h_3^{HARA} of Lie algebra L_3^{HARA} listed in Table 6.1 and defined as follows

$$h_3^{HARA} =< e_3 >= \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)\frac{\partial}{\partial V}\right\rangle.$$

If $\gamma \to 0$ this subalgebra transforms into h_1^{LOG} listed in Table 7.1 with parameter $\beta = 0$. This subalgebra of Lie algebra L_3^{LOG} is defined as

$$\begin{aligned} h_1^{LOG} &= < e_1 \cos\beta + e_3 \sin\beta > \\ &= \left\langle \cos\beta l \frac{\partial}{\partial l} + \cos\beta h \frac{\partial}{\partial h} + \left(\sin\beta - \cos\beta \int \overline{\Phi}(t)dt\right) \frac{\partial}{\partial V} \right\rangle. \end{aligned}$$

Indeed, if $\beta = 0$ then

$$h_3^{HARA} \underset{\gamma\to 0}{\longrightarrow} \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} - \int \overline{\Phi}(t)dt \frac{\partial}{\partial V} \right\rangle = h_1^{LOG}|_{\beta=0}$$

The standard invariants that we later use to obtain the reduction are defined as (6.5) and (6.6) for the case of HARA utility. Let us list them here for the convenience of the reader

$$\begin{aligned} inv_1^{HARA} &= t, \quad inv_2^{HARA} = z = \frac{l}{h}, \\ inv_3^{HARA} &= W^{HARA}(z,t) = h^{-\gamma} V^{HARA}(l,h,t) - \frac{1-\gamma}{\gamma} h^{-\gamma} \int \overline{\Phi}(t)dt. \end{aligned}$$

The standard invariants for the case with logarithmic utility are defined in (7.6) and (7.7) and look as follows

$$\begin{aligned} inv_1^{LOG} &= t, \quad inv_2^{LOG} = z = \frac{l}{h} \\ inv_3^{LOG} &= W^{LOG}(z,t) = V^{LOG} + \log h \left(\tan\beta + \int \overline{\Phi}(t)dt\right). \end{aligned}$$

The reader can see that the correspondence between the case of HARA utility and the case of logarithmic one is not satisfied anymore. Due to the certain freedom that we have in the choice of invariants we can chose the invariants of h_3^{HARA} in a different way. Indeed, if instead of $W^{HARA}(z,t)$ we choose $\tilde{W}^{HARA}(z,t)$ defined as

$$\tilde{W}^{HARA}(z,t) = W^{HARA}(z,t)(1-\gamma)^{\gamma} + \frac{1-\gamma}{\gamma} \int \overline{\Phi}(t)dt,$$

and use inv_1^{HARA} and inv_2^{HARA} as new independent variables t and z, whereas $\tilde{W}^{HARA}(z,t)$ as a new dependent variable then substituting these new variables into equation (5.20) instead of equation (6.7) we would obtain

$$
\begin{aligned}
& \tilde{W}_t(z,t) + \frac{1}{2}\eta^2\left(2z(1-\gamma)\tilde{W}_z(z,t) + z^2\tilde{W}_{zz}(z,t)\right) \\
+ \quad & (rz+\delta)\tilde{W}_z(z,t) - (\mu-\delta)z\tilde{W}_z(z,t) \\
- \quad & \frac{(\alpha-r)^2\tilde{W}_z^2(t,z) - 2(\alpha-r)\eta\rho\tilde{W}_z(t,z)((1-\gamma)\tilde{W}_z(t,z) + z\tilde{W}_{zz}(t,z))}{2\sigma^2\tilde{W}_{zz}(t,z)} \\
- \quad & \frac{\eta^2\rho^2\sigma^2((1-\gamma)\tilde{W}_z(t,z) + z\tilde{W}_{zz}(t,z))^2}{2\sigma^2\tilde{W}_{zz}(t,z)} \\
- \quad & \gamma\left(\frac{1}{2}\eta^2(1-\gamma) - \mu + \delta\right)\tilde{W}(z,t) \\
- \quad & (1-\gamma)\left(\frac{1}{2}\eta^2(1-\gamma) - \mu + \delta\right)\int\overline{\Phi}(t)dt \\
- \quad & \frac{1-\gamma}{\gamma}\overline{\Phi}(t) + \frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}}\tilde{W}_z^{-\frac{\gamma}{1-\gamma}}(t,z) = 0.
\end{aligned} \tag{5.48}
$$

If we use invariants inv_1^{LOG} and inv_2^{LOG} (7.6) as new independent variables t and z and invariant $W^{LOG}(z,t)$ (7.7) as a new dependent variable and substite them into (5.35) we obtain

$$
\begin{aligned}
& W_t(z,t) + \frac{1}{2}\eta^2\left(2zW_z(z,t) + z^2W_{zz}(z,t)\right) \\
+ \quad & (rz+\delta)W_z(z,t) - (\mu-\delta)zW_z(z,t) \\
- \quad & \frac{(\alpha-r)^2W_z^2(z,t) - 2(\alpha-r)\eta\rho W_z(z,t)(W_z(z,t) + zW_{zz}(z,t))}{2\sigma^2W_{zz}(z,t)} \\
- \quad & \frac{\eta^2\rho^2\sigma^2(W_z(z,t) + zW_{zz}(z,t))^2}{2\sigma^2W_{zz}(z,t)} \\
- \quad & \left(\frac{1}{2}\eta^2 - \mu + \delta\right)\int\overline{\Phi}(t)dt - \overline{\Phi}(t)(\log W_z(z,t) - \log\overline{\Phi}(t) + 1) = 0.
\end{aligned}
$$

This equation corresponds to equation (7.8) with $\beta = 0$ and the reader can see that this equation is a formal limit of (5.48) as $\gamma \to 0$, indeed. However, equation (5.48) is even more voluminous than equation (6.7) that is obtained through standard invariants, therefore further we use a standard reduction procedure in order to get a convenient form of equations in both cases with HARA and logarithmic utility and do not pay particular attention to the correspondence between the two. It is possible to obtain invariants that would preserve that relation even for the equations of lesser dimensions but we decided to provide the study of both cases in their own right to get the most convenient form of reduced equations that could be used for further solutions.

5.4 Results of the chapter

In this chapter we have introduced the notations needed for Lie group analysis and carried out a complete Lie group analysis for the optimization problem with two different utility functions, i.e. for two different three dimensional PDEs (5.20) and (5.35) which contain an arbitrary function $\overline{\Phi}(t)$. In both cases we are able to solve these rather voluminous problems and find the admitted Lie algebras L_3^{HARA} and L_3^{LOG}. The study of the three dimensional problems is connected with a lot of tedious calculations, even the first step on which one needs to find the determining system is, in fact, non-trivial. These difficulties become even more evident if we have an arbitrary function in the studied equation. The problem becomes slightly more tractable if one applies package **IntroToSymmetry**, but the majority of the calculations still are to be done manually. In this way we get the system of partial differential equations containing 137 and 130 different equations correspondingly. These equations define the generators of the corresponding algebras. Computer systems that we know of, unfortunately, can not solve the obtained system of differential equations. This has to be a step-by-step handmade procedure. To our knowledge, this is a first application of Lie group analysis to such problems for a general liquidation time distribution. If we look on the amount of calculations it is also understandable why just few cases of three dimensional PDEs are profoundly studied in the literature.

We also show that if and only if the liquidation time defined by a survival function $\overline{\Phi}(t)$ is distributed exponentially, then for both types of the utility functions we get an additional symmetry. We prove that both Lie algebras admit this extension, i.e. we obtain the four dimensional L_4^{HARA} and L_4^{LOG} correspondingly for the case of exponentially distributed liquidation time. Indeed, the case of exponentially distributed liquidation time is actually similar to the infinite-horizon random income problem and several other models studied in the literature (see, for instance, [22]), yet our work is the first to our knowledge that explicitly shows which properties make an exponentially distributed liquidation time a distinguished case, that allows a reduction of an original three dimensional PDEs to

ODEs. This is a very important result, since a lot of the works in the field use similar reductions and do not mention that there is actually no other distribution that allows a Lie type reduction of a PDE to an ODE. With a help of Lie group analysis we explain what makes exponential liquidation time distribution a distinguished case, the only situation when one can reduce the three dimensional HJBs to ODEs.

We have also shown a connection between HARA and logarithmic utility problems that hold on the level of HJB equations and on the level of algebraic structures standing behind the equations.

In the next chapters we provide reductions of (5.20) and (5.35) using the admitted Lie algebras.

6

Reductions of PDEs with HARA utility

In this chapter we discuss the reductions of the equations with HARA utility function that follow from the results of Lie group analysis of the problem that was carried out in the previous chapter. Let us discuss the complete set of possible reductions of three dimensional PDEs (5.20) arising in the case of HARA-utility function. In the previous chapter we have seen that some of the generators of the admitted Lie algebras L_3^{HARA} and L_4^{HARA} ($L_{3,4}^{HARA}$) described in Theorem 12 and L_3^{LOG} and L_4^{LOG} ($L_{3,4}^{LOG}$) as presented in Theorem 13 coincide in all studied cases. Let us now briefly discuss the mathematical and economic meaning of these generators.

The first generator $\mathbf{U}_1 = \frac{\partial}{\partial V}$ means that the original value function $V(t, l, h)$, solution of (5.20) for HARA utility or (5.35) for log utility correspondingly, can be shifted on any constant and still be a solution of the main equation. Neither allocation π (5.21) or consumption function c (5.22) will change their values, because they depend only on the derivatives of the value functions. In some sense it is a trivial symmetry, since the equation (5.20) contains just the derivatives of $V(l, h, t)$ we certainly can add a constant to this function and it still will be a solution of the equation. This symmetry does not give a rise to any reductions of the studied three dimensional PDEs.

The second generator $\mathbf{U}_2 = e^{rt}\frac{\partial}{\partial l}$ means that the value of the independent variable l can be shifted on the arbitrary value ae^{rt}, i.e. the shift $l \to l + ae^{rt}$,

$a - const.$ leaves the solution unaltered. From economical point of view it means that we can arbitrary shift the initial liquidity on a bank account $a, a > 0$ or credit $a, a < 0$. The value function $V(l,h,t)$ and the allocation-consumption strategy (π, c) will be unaltered, see (5.20) or (5.35). This symmetry is also trivial since it does not give any reductions of the original three dimensional PDEs.

Furthermore we also get an infinitely-dimensional algebra $L_\infty =< \psi(h,t)\frac{\partial}{\partial V} >$ where the function $\psi(h,t)$ is any solution of the linear PDE

$$\psi_t(h,t) + \frac{1}{2}\eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta)h\psi_h(h,t) = 0, \tag{6.1}$$

and has a very interesting meaning. We can add any solution $\psi(h,t)$ of this equation to the value function $V(l,h,t)$ without any changes of the allocation-consumption strategy (π, c). In economical sense it means that the additional use of some financial instrument which is the solution of $\psi_t(h,t) + \frac{1}{2}\eta^2 h^2 \psi_{hh}(h,t) + (\mu - \delta)h\psi_h(h,t) = 0$, i.e. a financial instrument which value is defined just by the paper value of the illiquid asset and time in accordance with (6.1), can not change the allocation-consumption strategy (π, c).
Now we are going to discuss the possible reductions of the three dimensional PDE (5.20) arising in the case of the HARA utility in detail.

6.1 Reductions in the case of general liquidation time distribution and HARA utility

In order to describe all non-equivalent invariant solutions to (5.20) we need to find an optimal system of subalgebras for the admitted Lie algebra (5.23) described by Theorem 12. Let us at first remind you how does our problem look in a general case

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl + \delta h)V_l(t,l,h) + (\mu - \delta)hV_h(t,l,h) \\ - & \frac{(\alpha - r)^2 V_l^2(t,l,h) + 2(\alpha - r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ + & \frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma}\overline{\Phi}(t) = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{6.2}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as in (5.21) and (5.22). The Lie algebra admitted by this PDE is described in Theorem 12 and is three or four dimensional depending on the properties of the function $\overline{\Phi}(t)$. The classification of all real three and four dimensional solvable Lie algebras is given in [50]. The authors provide optimal systems of subalgebras for every real solvable three and four dimensional Lie algebra. In this Section we study a three dimensional case. In order to have a consistency with this classification we change a basis of the corresponding algebra to a suitable one for every Lie algebras described above. One can also use the software package **SymboLie** [45] (a supplement package for **Mathematica**) to find an optimal system of subalgebras for a given Lie algebra directly, but we prefer to use the classification and notation provided in [50] for the sake of consistency.

The reductions can be obtained if we replace original variables with new independent and dependent variables which are invariant under the action of the Lie group or subgroup of this Lie group admitted by the equation. In this section we will list all non-equivalent reductions and provide all possible reduced equations.

As we mentioned before we use just the admitted Lie algebras to get the invariants of the corresponding groups. We do not need to provide the explicit form if the groups and subgroups.

First of all we should introduce the notations that we use further.

We denote by h_i the subalgebras of the Lie algebra L_3^{HARA} (5.23) (or L_4^{HARA} correspondingly) and H_i for the subgroups of the group G_3^{HARA} (or G_4^{HARA}) which are generated with the help of the exponential map by h_i.

6.1.1 System of optimal subalgebras of L_3^{HARA}

At first let us reassign the basis of L_3^{HARA} to adopt the real three dimensional Lie algebra L_3^{HARA} to the classification obtained in [50] in the following way $\mathbf{U}_1 = e_2, \mathbf{U}_2 = e_1, \mathbf{U}_3 = e_3$. The basis is now defined as

$$e_1 = e^{rt}\frac{\partial}{\partial l},\ e_2 = \frac{\partial}{\partial V},\ e_3 = l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)\frac{\partial}{\partial V}. \quad (6.3)$$

In this basis algebra L_3^{HARA} has two non zero commutation relations which are given now in the following form

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = \gamma e_2. \quad (6.4)$$

Now we can see that $L_3^{HARA} = \langle e_1, e_2, e_3 \rangle$ corresponds to $A_{3,5}^{\gamma}$, where $0 < \gamma < 1$, in the classification of [50]. The system of optimal subalgebras for this algebra is listed in Table 6.1.

Dimension of the subalgebra	System of optimal subalgebras of the Lie algebra L_3^{HARA} (5.23)
1	$h_1 = \langle e_1 \rangle\,,\ h_2 = \langle e_2 \rangle\,,\ h_3 = \langle e_3 \rangle\,,\ h_4 = \langle\ e_1 \pm e_2 \rangle$
2	$h_5 = \langle e_1, e_2 \rangle\,,\ h_6 = \langle e_3, e_1 \rangle\,,\ h_7 = \langle e_3, e_2 \rangle$

Table 6.1: [50] The optimal system of one and two dimensional subalgebras of L_3^{HARA} (5.23).

The optimal system of one- and two- parameter subalgebras give rise to the system of one or two dimensional symmetry subgroups H_i of the studied PDE. Our goal now is to find all possible corresponding reductions and to describe the solutions which are invariant under the action of the groups H_i.

6.1.2 One-dimensional subalgebras of L_3^{HARA} and corresponding reductions

Let us look closer at all one dimensional subalgebras listed in the first row of Table 6.1. If we try to reduce the tree dimensional PDE to a two dimensional one, then such reduction can be provided by one of the corresponding one dimensional subgroups if at all.

L_3^{HARA} has four one-dimensional subalgebras h_i which give rise to four one parameter subgroups H_i. Our goal is to study the corresponding invariant solutions. Not every subgroup out of all listed in Table 6.1 provides a nontrivial reduction of the original PDE, but if a non-trivial reduction of Lie type exists, then it can be found out through a suitable subalgebra listed in Table 6.1. We have already discussed the meaning of h_1 and h_2 above in Section 6.1. These two cases do not give us any reductions, as we have already discussed these two cases in the beginning go this section. Let us start with the next case.

Case $H_3(h_3)$. Under h_3 in Table 6.1 we denote the subalgebra that is defined as

$$h_3 =< e_3 >= \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)\frac{\partial}{\partial V}\right\rangle.$$

H_3 denotes a corresponding subgroup. To find the invariants of H_3 we solve a characteristic system of the equations

$$\frac{dt}{0} = \frac{dl}{l} = \frac{dh}{h} = \frac{dV}{\left(\gamma V - (1-\gamma)\int \overline{\Phi}(t)dt\right)},$$

where the first equation of the system is a formal notation that shows that independent variable t is actually an invariant of the equation under the action of H_3. We can obtain two other independent invariants solving the system above

$$\begin{aligned} inv_1 &= t, \quad inv_2 = z = \frac{l}{h}, && (6.5)\\ inv_3 &= W(z,t) = h^{-\gamma}V(l,h,t) - \frac{1-\gamma}{\gamma}h^{-\gamma}\int \overline{\Phi}(t)dt. && (6.6)\end{aligned}$$

These invariants (6.5) can be used as new independent variables t, z and the invariant (6.6) as the new dependent variable $W(t,z)$ to reduce the three dimensional PDE (6.2) to a two dimensional one

$$\begin{aligned} & W_t(t,z) + \frac{1}{2}\eta^2\left(\gamma(\gamma-1)W(t,z) - 2(\gamma-1)zW_z(t,z) + z^2W_{zz}(t,z)\right)\\ + & (rz+\delta)W_z(t,z) + (\mu-\delta)(\gamma W(t,z) - zW_z(t,z))\\ - & \frac{(\alpha-r)^2W_z^2(t,z) - 2(\alpha-r)\eta\rho W_z(t,z)((1-\gamma)W_z(t,z) + zW_{zz}(t,z))}{2\sigma^2W_{zz}(t,z)}\\ - & \frac{\eta^2\rho^2\sigma^2((1-\gamma)W_z(t,z) + zW_{zz}(t,z))^2}{2\sigma^2W_{zz}(t,z)}\\ + & \frac{(1-\gamma)^2}{\gamma}\overline{\Phi}(t)^{\frac{1}{1-\gamma}}W_z^{-\frac{\gamma}{1-\gamma}}(t,z) = 0, \qquad W \underset{t\to\infty}{\longrightarrow} 0. \qquad (6.7)\end{aligned}$$

After this reduction the allocation $\pi(t,z,h)$ and consumption $c(t,z,h)$ strategies (5.21) and (5.22) look as follows

$$\begin{aligned} \pi(t,z,h) &= h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma(1-\gamma) - \alpha + r}{\sigma^2}\frac{W_z(t,z)}{W_{zz}(t,z)}\right), && (6.8)\\ c(t,z,h) &= h(1-\gamma)W_z^{-\frac{1}{1-\gamma}}(t,z)\overline{\Phi}(t)^{\frac{1}{1-\gamma}}. && (6.9)\end{aligned}$$

Case $H_4(h_4)$. This subalgebra h_4 is spanned by the generator $e^{rt}\frac{\partial}{\partial l} \pm \frac{\partial}{\partial V}$. We can write a characteristic system for this case

$$\frac{dl}{e^{rt}} = \frac{dh}{0} = \frac{dt}{0} = \frac{dV}{\mp 1}. \qquad (6.10)$$

We can see from this system that two independent variables t and h are invariants. The third invariant is $W(t,h) = V(l,h,t) \mp e^{-rt}l$. This means that in this case the value function has the form $V = W(t,h) \pm e^{-rt}l$. This essentially means that V in this case is a linear function of l. From Chapter 3 we already know that the value function described by (3.10) should be concave in l, so this case is not interesting for our problem from the economical point of view. Since though H_4 gives us a reduction of the equation (5.20) the corresponding value function does not satisfy the conditions sufficient for a solution of (3.10) in the class of l-concave functions.

The two-dimensional subalgebras of L_3^{HARA} do not give us any meaningful substitutions, what would be able to reduce the problem to an ODE. This means that if we deal with a HARA utility function and a general form of the liquidation time distribution we have just one possibility to reduce the three dimensional PDE (6.2) to a two dimensional one (6.7) using the substitution (6.5)-(6.6) or equivalent. Any further reductions in the framework of Lie group analysis are not possible. It does not mean that any other simplifications of the PDE are not possible, but we do not have an algorithmic way to obtain them. At the same time it is important to note that if we study problem (5.20) we can for sure apply numeric or quantitate methods to study the equation (6.7), which is simpler than the original one.

6.2 A special case of an exponential liquidation time distribution with HARA utility function

The exponential liquidation time distribution with the survival function $\overline{\Phi}(t) = e^{-\kappa t}$ is a special case of the problem that deserves our separate attention. We have proven in Theorem 12 that in this case and just in this case we obtain an extended four dimensional Lie algebra and can hope to obtain deeper reductions than in the general case. Inserting this special form of $\overline{\Phi}(t)$ into (5.20) we obtain the following equation

$$\begin{aligned}
& V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\
- & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\
+ & \frac{(1-\gamma)^2}{\gamma} e^{-\frac{\kappa t}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma} e^{-\kappa t} = 0, \qquad V \underset{t\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{6.11}$$

This equation admits Lie algebra L_4^{HARA} spanned by the generators (5.23) and (5.24) as we have demonstrated in Theorem 12.

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function V

$$\pi(t,l,h) = -\frac{\eta\rho\sigma h V_{lh}(t,l,h) + (\alpha-r)V_l(t,l,h)}{\sigma^2 V_{ll}(t,l,h)}, \tag{6.12}$$

$$c(t,l,h) = (1-\gamma)V_l(t,l,h)^{-\frac{1}{1-\gamma}} e^{\frac{-\kappa t}{1-\gamma}}. \tag{6.13}$$

Since the non-zero commutators of L_4^{HARA} depend on the parameters κ, r and γ the inner structure of the Lie algebra L_4^{HARA} is different for different values of these parameters. If we further use the classification of all solvable real three and four dimensional Lie algebras proposed in [50], we can see that depending on the relations between the parameters of the equation we obtain two different algebraic structures. Using the notation of [50] we see that when $\kappa \neq r\gamma$ then L_4^{HARA} corresponds to $2A_2$, whereas when $\kappa = r\gamma$ then L_4^{HARA} corresponds to $A_{3,5}^{\gamma} \bigoplus A_1$ (see [50]). We now look at each of these cases separately.

6.2.1 System of optimal subalgebras of L_4^{HARA} for the case $\kappa \neq r\gamma$

Let us at first regard a situation, when $\kappa \neq r\gamma$. To make the structure of L_4^{HARA} visible and comparable with the notation in [50] we transform the basis of L_4^{HARA} as follows

$$\begin{aligned}
e_1 &= \frac{r\mathbf{U}_3 + \mathbf{U}_4}{\kappa - r\gamma}, \quad e_2 = \mathbf{U}_1 \\
e_3 &= -\frac{\kappa\mathbf{U}_3 + \gamma\mathbf{U}_4}{\kappa - r\gamma}, \quad e_4 = \mathbf{U}_2.
\end{aligned}$$

Now the generators of $L_4^{HARA} = \langle e_1, e_2, e_3, e_4\rangle$ have a form

$$
\begin{aligned}
e_1 &= \frac{r}{\kappa - r\gamma} l \frac{\partial}{\partial l} + \frac{r}{\kappa - r\gamma} h \frac{\partial}{\partial h} + \frac{1}{\kappa - r\gamma}\frac{\partial}{\partial t} - \left(V - \frac{(1-\gamma)r}{\kappa(\kappa - r\gamma)} e^{-\kappa t}\right)\frac{\partial}{\partial V},\\
e_2 &= \frac{\partial}{\partial V},\\
e_3 &= -\frac{\kappa}{\kappa - r\gamma} l \frac{\partial}{\partial l} - \frac{\kappa}{\kappa - r\gamma} h \frac{\partial}{\partial h} - \frac{\gamma}{\kappa - r\gamma}\frac{\partial}{\partial t} - \frac{(1-\gamma)}{\kappa - r\gamma} e^{-\kappa t}\frac{\partial}{\partial V},\\
e_4 &= e^{rt}\frac{\partial}{\partial l},
\end{aligned}
$$

where $\kappa \neq r\gamma$. In this basis the Lie algebra L_4^{HARA} has only two non-zero commutation relations

$$[e_1, e_2] = e_2, \quad [e_3, e_4] = e_4. \tag{6.14}$$

We can see that L_4^{HARA} corresponds to $2A_2$ or in another common notation $A_2 \oplus A_2$, according to the classification [50]. The system of optimal subalgebras for an algebra of this type is listed in Table 6.2. We use this optimal system of

Dimension of the subalgebra	System of optimal subalgebras of algebra L_4^{HARA}, $\kappa \neq r\gamma$
1	$h_1 = \langle e_2\rangle$, $h_2 = \langle e_3\rangle$, $h_3 = \langle e_4\rangle$, $h_4 = \langle e_1 + \omega e_3\rangle$, $h_5 = \langle e_1 \pm e_4\rangle$, $h_6 = \langle e_2 \pm e_4\rangle$, $h_7 = \langle e_2 \pm e_3\rangle$
2	$h_8 = \langle e_1, e_3\rangle$, $h_9 = \langle e_1, e_4\rangle$, $h_{10} = \langle e_2, e_3\rangle$, $h_{11} = \langle e_2, e_4\rangle$, $h_{12} = \langle e_1 + \omega e_3, e_2\rangle$, $h_{13} = \langle e_3 + \omega e_1, e_4\rangle$, $h_{14} = \langle e_1 \pm e_4, e_2\rangle$, $h_{15} = \langle e_3 \pm e_2, e_4\rangle$, $h_{16} = \langle e_1 + e_3, e_2 \pm e_4\rangle$
3	$h_{17} = \langle e_1, e_3, e_2\rangle$, $h_{18} = \langle e_1, e_4, e_2\rangle$, $h_{19} = \langle e_1, e_3, e_4\rangle$, $h_{20} = \langle e_2, e_3, e_4\rangle$, $h_{21} = \langle e_1 \pm e_3, e_2, e_4\rangle$, $h_{22} = \langle e_1 + \omega e_3, e_2, e_4\rangle$

Table 6.2: [50] The optimal system of one, two and three dimensional subalgebras of L_4^{HARA} for $\kappa \neq r\gamma$. Here ω is a parameter, $-\infty < \omega < \infty$.

subalgebras to obtain all non equivalent reductions and list them in the next section.

6.2.2 One-dimensional subalgebras of L_4^{HARA}, $\kappa \neq r\gamma$, and corresponding reductions

L_4^{HARA} in the case $\kappa \neq r\gamma$ has ten one-dimensional subalgebras listed in the first row of Table 6.2, but by far not all of them can provide meaningful reductions of (6.11). Before we start a step-by-step discussion regarding each subalgebra in the optimal system of subalgebras listed in Table 6.2 we should remind the reader that we have already discussed two of them before. In Section 6.1.2 we have already shown that subalgebras $h_1 = \left\langle \frac{\partial}{\partial V} \right\rangle$ and $h_3 = \left\langle e^{rt}\frac{\partial}{\partial l} \right\rangle$ and $h_6 = \left\langle \frac{\partial}{\partial V} \pm e^{rt}\frac{\partial}{\partial l} \right\rangle$ describe important invariant properties of the equation (6.11) but they do not provide any new meaningful reductions.

Now we are going to regard other subalgebras in detail.

Case $H_2(h_2)$. This subalgebra is spanned by a generator e_3

$$\begin{aligned} h_2 =< e_3 > \quad &= \quad \left\langle -\frac{\kappa}{\kappa - r\gamma} l \frac{\partial}{\partial l} - \frac{\kappa}{\kappa - r\gamma} h \frac{\partial}{\partial h} - \frac{\gamma}{\kappa - r\gamma}\frac{\partial}{\partial t} \right. \\ &- \left. \frac{(1-\gamma)}{\kappa - r\gamma} e^{-\kappa t} \frac{\partial}{\partial V} \right\rangle . \end{aligned} \tag{6.15}$$

As in the case L_3^{HARA} we can find invariants of the corresponding subgroup H_4 solving the characteristic system

$$\frac{dl}{\kappa l} = \frac{dh}{\kappa h} = \frac{dt}{\gamma} = \frac{e^{\kappa t} dV}{1-\gamma}.$$

Out of the characteristic system we find the following invariants

$$inv_1 \quad = \quad z = \frac{l}{h}, \quad inv_2 = \tau = \frac{\kappa}{\gamma} t - \log h, \tag{6.16}$$

$$inv_3 \quad = \quad W(z,\tau) = V + \frac{1-\gamma}{\gamma\kappa} e^{-\kappa t}. \tag{6.17}$$

Substituting these new independent variables z and τ and a new dependent variable $W(z,\tau)$ into (6.11) we obtain a two-dimensional equation

$$\begin{aligned}
& \frac{\kappa}{\gamma}W_\tau(z,\tau)+\frac{1}{2}\eta^2\left(2zW_z(z,\tau)+z^2W_{zz}(z,\tau)\right)+(rz+\delta)W_z(z,\tau)\\
- & (\mu-\delta)zW_z(z,\tau)-\frac{(\alpha-r)^2W_z^2(z,\tau)}{2\sigma^2W_{zz}(z,\tau)}\\
- & \frac{2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau)+zW_{zz}(z,\tau))+\eta^2\rho^2\sigma^2(W_z(z,\tau)+zW_{zz}(z,\tau))^2}{2\sigma^2W_{zz}(z,\tau)}\\
+ & \frac{(1-\gamma)^2}{\gamma}e^{-\frac{\gamma}{1-\gamma}\tau}W_z^{-\frac{\gamma}{1-\gamma}}=0, \qquad W\underset{\tau\to\infty}{\longrightarrow}0.
\end{aligned} \tag{6.18}$$

Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the function $W(z,\tau)$

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z+\frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), \tag{6.19}$$

$$c(z,\tau,h) = h(1-\gamma)W_z(z,\tau)^{-\frac{1}{1-\gamma}}e^{-\frac{\gamma\tau}{1-\gamma}}. \tag{6.20}$$

Case $H_4(h_4)$. This sub algebra h_4 is spanned by the generator $e_1+\omega e_3$

$$\begin{aligned}
h_4 &= < e_1+\omega e_3 > \\
&= \left\langle \frac{r-\omega\kappa}{\kappa-r\gamma}l\frac{\partial}{\partial l}+\frac{r-\omega\kappa}{\kappa-r\gamma}h\frac{\partial}{\partial h}+\frac{1-\omega\gamma}{\kappa-r\gamma}\frac{\partial}{\partial t}-\left(V-\frac{(1-\gamma)(r-\omega\kappa)}{\kappa(\kappa-r\gamma)}e^{-\kappa t}\right)\frac{\partial}{\partial V}\right\rangle.
\end{aligned} \tag{6.21}$$

Since parameter ω can have any value due to the interplay of the constants we need to regard three cases separately.

First, if $\omega=r/\kappa$ this case is defined by a generator

$$h_4=\left\langle e_1+\frac{r}{\kappa}e_3\right\rangle=\left\langle\frac{1}{\kappa}\frac{\partial}{\partial t}-V\frac{\partial}{\partial V}\right\rangle.$$

The invariants of the corresponding subgroup H_4 for this case are as follows

$$inv_1 = l, \quad inv_2=h, \tag{6.22}$$

$$inv_3 = W(l,h)=Ve^{\kappa t}. \tag{6.23}$$

Using two invariants (6.22) as the new independent variables and (6.23) as the dependent variable in (6.11) we obtain a two dimensional PDE

$$\begin{aligned}
- & \kappa W(l,h)+\frac{1}{2}\eta^2h^2W_{hh}(l,h)+(rl+\delta h)W_l(l,h)+(\mu-\delta)hW_h(l,h)\\
- & \frac{(\alpha-r)^2W_l^2(l,h)+2(\alpha-r)\eta\rho hW_l(l,h)W_{lh}(l,h)+\eta^2\rho^2\sigma^2h^2{W_{lh}}^2(l,h)}{2\sigma^2W_{ll}(l,h)}\\
+ & \frac{(1-\gamma)^2}{\gamma}W_l(l,h)^{-\frac{\gamma}{1-\gamma}}-\frac{1-\gamma}{\gamma}=0.
\end{aligned} \tag{6.24}$$

It also means that we have a value function $V(l,h,t)=e^{-\kappa t}W(l,h)$, where $W(l,h)$ satisfies (6.24) and the time dependence of the value function $V(l,h,t)$ is defined completely by the factor $e^{-\kappa t}$ and condition $V \underset{t\to\infty}{\longrightarrow} 0$ will be satisfied for any $W(l,h)$. Here the investment $\pi(l,h)$ and consumption $c(l,h)$ look as follows in terms of the function W

$$\pi(l,h) = -\frac{\eta\rho\sigma h W_{lh}(l,h)+(\alpha-r)W_l(l,h)}{\sigma^2 W_{ll}(t,l,h)}, \tag{6.25}$$

$$c(l,h) = (1-\gamma)W_l(l,h)^{-\frac{1}{1-\gamma}}. \tag{6.26}$$

The second case can be obtained if $\omega=\frac{1}{\gamma}$. In this case the algebra h_4 is spanned by the generator

$$h_4=\left\langle e_1+\frac{1}{\kappa}e_3\right\rangle=\left\langle -\frac{1}{\gamma}l\frac{\partial}{\partial l}-\frac{1}{\gamma}h\frac{\partial}{\partial h}-\left(V+\frac{1-\gamma}{\kappa\gamma}e^{-\kappa t}\right)\frac{\partial}{\partial V}\right\rangle. \tag{6.27}$$

One can find the following invariants of the subgroup H_4

$$inv_1 = z=\frac{l}{h}, \quad inv_2=t, \tag{6.28}$$

$$inv_3 = W(z,t)=h^{-\gamma}V+\frac{1-\gamma}{\gamma\kappa}h^{-\gamma}e^{-\kappa t}. \tag{6.29}$$

Substituting the invariants z and t (6.28) as independent variables and the invariant $W(z,t)$ (6.29) as the dependent variable into (6.11) we derive the following two dimensional equation

$$\begin{aligned}
& W_t(t,z)+\frac{1}{2}\eta^2\left(\gamma(\gamma-1)W(t,z)-2(\gamma-1)zW_z(t,z)+z^2W_{zz}(t,z)\right)\\
+\ & (rz+\delta)W_z(t,z)+(\mu-\delta)(W(t,z)-zW_z(t,z))\\
-\ & \frac{(\alpha-r)^2W_z^2(t,z)-2(\alpha-r)\eta\rho W_z(t,z)((1-\gamma)W_z(t,z)+zW_{zz}(t,z))}{2\sigma^2W_{zz}(t,z)}\\
-\ & \frac{\eta^2\rho^2\sigma^2((1-\gamma)W_z(t,z)+zW_{zz}(t,z))^2}{2\sigma^2W_{zz}(t,z)}\\
+\ & \frac{(1-\gamma)^2}{\gamma}e^{-\frac{\kappa t}{1-\gamma}}W_z^{-\frac{\gamma}{1-\gamma}}=0, \quad W\underset{t\to\infty}{\longrightarrow}0.
\end{aligned} \tag{6.30}$$

Here the investment $\pi(t,z,h)$ and consumption $c(t,z,h)$ look as follows in terms of the function W

$$\pi(t,z,h) = h\left(\frac{\eta\rho}{\sigma}z+\frac{\eta\rho\sigma(1-\gamma)-\alpha+r}{\sigma^2}\frac{W_z(t,z)}{W_{zz}(t,z)}\right), \tag{6.31}$$

$$c(t,z,h) = h(1-\gamma)e^{\frac{-\kappa t}{1-\gamma}}W_z(t,z)^{-\frac{1}{1-\gamma}}. \tag{6.32}$$

Before we move on, we need to point out two facts about this substitution. First of all, it directly corresponds to the substitution (6.5) that we have found earlier for a case of general liquidation time distribution. Second notion that we need to make is that an analogous symmetry is used in [55]. The framework of the problem is a bit different, since authors regard a fixed pre-determined liquidation time, yet the substitution they use to reduce a three dimensional PDE to a two dimensional one is very similar. The authors do not carry out a complete analysis of their problem and have to work with an equation of the second order. This makes a problem significantly more complicated, yet in their framework a Lie type reduction to a one dimensional equation is possible, as it was shown in Chapter 4. The authors in [22] work with the problem of random income in an infinite time set-up, so their problem is two-dimensional by design. They also use a similar substitution to reduce their two dimensional problem to a one dimensional case. Yet it is clear that the substitution they use corresponds to this one. Now, since we have carried out a complete analysis of all Lie type substitutions we see which of them were explored in the literature before and we give a solid mathematical explanation of this substitutions instead of an educated guess that is most commonly used to simplify the problems of such type.

These two cases $\omega = r/\kappa$ and $\omega = \frac{1}{\gamma}$ are special since the generators, that span h_4, differ significantly form the generator in the most general case (6.21), when $\omega \neq \frac{r}{\kappa}, \frac{1}{\gamma}$.

The invariants in this more general case are as follows

$$
\begin{aligned}
inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = \frac{r - \omega\kappa}{1 - \omega\gamma} t - \log h, && (6.33)\\
inv_3 &= W(z,\tau) = V e^{\frac{\kappa - r\gamma}{1-\omega\gamma} t} + \frac{1-\gamma}{\gamma\kappa} e^{\frac{\gamma(\omega\kappa - r)}{1-\omega\gamma} t}, \quad \omega \neq \frac{r}{\kappa}, \frac{1}{\gamma}. && (6.34)
\end{aligned}
$$

As in previous cases we chose the first two invariants as independent variables and the last invariant $W(z,\tau)$ defined as (6.34) as a new dependent variable. Substituting these variables into (6.11) we obtain the reduced two dimensional equation

$$\begin{aligned}
& -\frac{\kappa - r\gamma}{1-\omega\gamma} W(z,\tau) + \frac{r-\omega\kappa}{1-\omega\gamma} W_\tau(z,\tau) + \frac{1}{2}\eta^2 \left(2zW_z(z,\tau) + z^2 W_{zz}(z,\tau)\right) \\
+ & \quad (rz+\delta)W_z(z,\tau) - (\mu-\delta)zW_z(z,\tau) \\
- & \quad \frac{(\alpha-r)^2 W_z^2(z,\tau) - 2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau) + zW_{zz}(z,\tau))}{2\sigma^2 W_{zz}(z,\tau)} \\
- & \quad \frac{\eta^2\rho^2\sigma^2(W_z(z,\tau)+zW_{zz}(z,\tau))^2}{2\sigma^2 W_{zz}(z,\tau)} + \frac{(1-\gamma)^2}{\gamma} W_z^{-\frac{\gamma}{1-\gamma}} = 0. \qquad (6.35)
\end{aligned}$$

Indeed, the condition $V \underset{t\to\infty}{\longrightarrow} 0$ is rewritten as $W \underset{\tau\to\infty}{\longrightarrow} 0$. Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the value function W

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), \qquad (6.36)$$

$$c(z,\tau,h) = (1-\gamma)he^{-\frac{\gamma\tau}{1-\gamma}} W_z^{-\frac{1}{1-\gamma}}. \qquad (6.37)$$

Case $H_5(h_5)$. This subalgebra is spanned by the following generator

$$\begin{aligned}
h_5 &= < e_1 \mp e_4 > \qquad (6.38) \\
&= \left\langle \left(\frac{r}{\kappa-r\gamma} l \pm e^{rt}\right)\frac{\partial}{\partial l} + \frac{r}{\kappa - r\gamma} h \frac{\partial}{\partial h} + \frac{1}{\kappa-r\gamma}\frac{\partial}{\partial t} \right. \\
&- \left. \left(V - \frac{(1-\gamma)r}{\kappa(\kappa-r\gamma)}e^{-\kappa t}\right)\frac{\partial}{\partial V}\right\rangle .
\end{aligned}$$

Solving the corresponding characteristic system we find the following invariants of the subgroup H_5

$$inv_1 = x = le^{-rt} \mp (\kappa - r\gamma)t, \quad inv_2 = y = he^{-rt}, \qquad (6.39)$$

$$inv_3 = W(x,y) = Ve^{(\kappa-r\gamma)t} - \frac{1-\gamma}{\gamma\kappa}e^{-r\gamma t}. \qquad (6.40)$$

Using expressions (6.39) as new independent variables z and τ and (6.40) as a new dependent variable $W(x,y)$ we reduce (6.11) to the two dimensional PDE

$$\begin{aligned}
& -(\kappa-r\gamma)W(x,y) + \frac{1}{2}\eta^2 y^2 W_{yy}(x,y) \\
+ & \quad (\delta y \mp (\kappa - r\gamma))W_x(x,y) + (\mu-\delta)yW_y(x,y) \\
- & \quad \frac{(\alpha-r)^2 W_x^2(x,y) + 2(\alpha-r)\eta\rho y W_x(x,y)W_{xy}(x,y) + \eta^2\rho^2\sigma^2 y^2 {W_{xy}}^2(x,y)}{2\sigma^2 W_{xx}(x,y)} \\
+ & \quad \frac{(1-\gamma)^2}{\gamma} W_x^{-\frac{\gamma}{1-\gamma}}(x,y) = 0, \quad W(x,0) \underset{x\to\mp\infty}{\longrightarrow} 0. \qquad (6.41)
\end{aligned}$$

Here the investment $\pi(x, y, h)$ and consumption $c(x, y, h)$ look as follows in terms of the value function $W(x, y)$

$$\pi(x,y,h) = -h\frac{\eta\rho\sigma W_{xy}(x,y) + (\alpha - r)y^{-1}W_x(x,y)}{\sigma^2 W_{xx}(x,y)}, \tag{6.42}$$

$$c(x,y,h) = h(1-\gamma)y^{-1}W_x^{-\frac{1}{1-\gamma}}. \tag{6.43}$$

Case $H_7(h_7)$. The last one dimensional subalgebra listed in Table 6.2 is subalgebra h_7 spanned by

$$h_7 = < e_2 \mp e_3 >= \left\langle \mp\frac{\kappa}{\kappa - r\gamma}l\frac{\partial}{\partial l} \mp \frac{\kappa}{\kappa - r\gamma}h\frac{\partial}{\partial h} \mp \frac{\gamma}{\kappa - r\gamma}\frac{\partial}{\partial t} - \left(1 \mp \frac{1-\gamma}{\kappa - r\gamma}e^{-\kappa t}\right)\frac{\partial}{\partial V}\right\rangle.$$

Solving the characteristic system we find the following invariants of the corresponding subgroup H_7

$$inv_1 = z = \frac{l}{h}, \quad inv_2 = \tau = \frac{\kappa}{\gamma}t - \log h, \tag{6.44}$$

$$inv_3 = W(z,\tau) = V \pm \frac{\kappa - r\gamma}{\gamma}t + \frac{1-\gamma}{\gamma\kappa}e^{-\kappa t}. \tag{6.45}$$

As before to get solutions of (6.11) invariant under the action of H_7 we use the invariants (6.44) as independent variables z, τ and the invariant (6.45) as the dependent variable $W(z, \tau)$. This way we reduce equation (6.11) to a two dimensional PDE that looks as follows

$$\begin{aligned} & \frac{\kappa}{\gamma}W_\tau(z,\tau) \pm \frac{\kappa - r\gamma}{\gamma} + \frac{1}{2}\eta^2\left(2zW_z(z,\tau) + z^2W_{zz}(z,\tau)\right) \\ + \; & (rz+\delta)W_z(z,\tau) - (\mu - \delta)zW_z(z,\tau) \\ - \; & \frac{(\alpha - r)^2W_z^2(z,\tau) - 2(\alpha - r)\eta\rho W_z(z,\tau)(W_z(z,\tau) + zW_{zz}(z,\tau))}{2\sigma^2W_{zz}(z,\tau)} \\ - \; & \frac{\eta^2\rho^2\sigma^2(W_z(z,\tau) + zW_{zz}(z,\tau))^2}{2\sigma^2W_{zz}(z,\tau)} + \frac{(1-\gamma)^2}{\gamma}e^{-\frac{\gamma}{1-\gamma}\tau}W_z^{-\frac{\gamma}{1-\gamma}} = 0, \; W(z,\tau) \underset{\tau\to\infty}{\longrightarrow} 0. \end{aligned} \tag{6.46}$$

Here the investment $\pi(z, \tau, h)$ and consumption $c(z, \tau, h)$ look as follows in terms of the function $W(z, \tau)$

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma - \alpha + r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), \tag{6.47}$$

$$c(z,\tau,h) = (1-\gamma)hW_z(z,\tau)^{-\frac{1}{1-\gamma}}e^{-\frac{\gamma}{1-\gamma}\tau}. \tag{6.48}$$

We studied in detail all seven one dimensional subalgebras from optimal system of subalgebras which can describe non equivalent invariant solutions of (6.2) in the case $\kappa \neq r\gamma$. We demonstrated that only in four cases meaningful reductions to a two dimensional PDEs are possible.

6.2.3 Two-dimensional subalgebras of L_4^{HARA}, $\kappa \neq r\gamma$, and corresponding reductions

As we studied one dimensional subalgebras we obtained the invariant solutions which are unaltered under the action of a one parameter group generated by one of the one dimensional subalgebras from the system of optimal subalgebras. Now we are going to find all non equivalent invariant solutions which are invariant under actions of two parameter subalgebras. This gives us a possibility to reduce a three dimensional PDE to an ODE.

In this section we go further and looking at the second row of the Table 6.2 to find the deeper reductions that can reduce PDE (6.2) to an ordinary differential equation.

Case $H_8(h_8)$. The first two dimensional sub algebra listed in Table 6.2 is subalgebra $h_8 =< e_1, e_3 >$ spanned by two generator defined as follows

$$\begin{aligned} e_1 &= \frac{r}{\kappa - r\gamma} l \frac{\partial}{\partial l} + \frac{r}{\kappa - r\gamma} h \frac{\partial}{\partial h} + \frac{1}{\kappa - r\gamma} \frac{\partial}{\partial t} - \left(V - \frac{(1-\gamma) r}{\kappa(\kappa - r\gamma)} e^{-\kappa t} \right) \frac{\partial}{\partial V}, \\ e_3 &= -\frac{\kappa}{\kappa - r\gamma} l \frac{\partial}{\partial l} - \frac{\kappa}{\kappa - r\gamma} h \frac{\partial}{\partial h} - \frac{\gamma}{\kappa - r\gamma} \frac{\partial}{\partial t} - \frac{(1-\gamma)}{\kappa - r\gamma} e^{-\kappa t} \frac{\partial}{\partial V}, \quad \kappa \neq r\gamma. \end{aligned}$$

The two dimensional subalgebra is spanned by two generators and each of them was actually studied before. We should find a simultaneous solution to both characteristic systems. We can use our previous knowledge and reformulate one of the characteristic systems in terms of invariant variables of the other one. This could be done in the following way. We have found a general form of the invariants of e_3 in case $H_2(h_2)$ (6.16), (6.17). Let us list them here again for the convenience of the reader

$$\begin{aligned} inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = \frac{\kappa}{\gamma} t - \log h, \\ inv_3 &= W(z, \tau) = V + \frac{1-\gamma}{\gamma\kappa} e^{-\kappa t}. \end{aligned}$$

If we rewrite the first generator of the subalgebra h_8 in terms of these three invariants z, τ and W as new independent and dependent variables correspondingly, we obtain

$$e_1 = \frac{1}{\gamma}\frac{\partial}{\partial\tau} - W\frac{\partial}{\partial W}. \tag{6.49}$$

Solving a corresponding characteristic system $\frac{d\tau}{1/\gamma} = \frac{dW}{-W}$ we obtain a new invariant

$$inv_{e_1} = Y(z) = W(z,\tau)e^{\gamma\tau}. \tag{6.50}$$

This way we obtain an invariant solution under the action of two parameter subgroup H_8. It means that we can take $Y(z)$ as a new dependent variable in (6.18) and z as a new independent one. Substituting these invariants into PDE (6.18) we obtain a new ODE

$$\begin{aligned}
&- \kappa Y(z) + \frac{1}{2}\eta^2\left(2zY_z(z) + z^2Y_{zz}(z)\right) + (rz+\delta)Y_z(z) - (\mu-\delta)zY_z(z)\\
&- \frac{(\alpha-r)^2Y_z^2(z) - 2(\alpha-r)\eta\rho Y_z(z)(Y_z(z)+zY_{zz}(z)) + \eta^2\rho^2\sigma^2(Y_z(z)+zY_{zz}(z))^2}{2\sigma^2Y_{zz}(z)}\\
&+ \frac{(1-\gamma)^2}{\gamma}Y_z^{-\frac{\gamma}{1-\gamma}}(z) = 0.
\end{aligned} \tag{6.51}$$

The condition $W(z,\tau) \underset{\tau\to\infty}{\longrightarrow} 0$ is satisfied for each finite solution $Y(z)$, because $W(z,\tau) = e^{-\gamma\tau}Y(z)$.

Naturally, the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ in terms of $Y(z)$ look now as follows

$$\begin{aligned}
\pi(z,\tau,h) &= h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{Y_z(z)}{Y_{zz}(z)}\right), && (6.52)\\
c(z,\tau,h) &= h(1-\gamma)Y_z(z)^{-\frac{1}{1-\gamma}}. && (6.53)
\end{aligned}$$

In terms of original variables t, l, h and $V(t,l,h)$ the substitution looks as follows

$$\begin{aligned}
z &= \frac{l}{h}, \quad \tau = \frac{\kappa}{\gamma}t - \log h, && (6.54)\\
Y(z) &= \left(V(t,l,h)e^{\kappa t} + \frac{1-\gamma}{\gamma\kappa}\right)h^{-\gamma}.
\end{aligned}$$

It also means that if we obtain a solution $Y(z)$ for (7.40) we obtain the value function that in terms of original variables looks like

$$V(t,l,h) = e^{-\kappa t}h^{\gamma}Y(l/h) - \frac{1-\gamma}{\gamma\kappa}e^{-\kappa t},$$

and the condition $V(l,h,t) \underset{t\to\infty}{\longrightarrow} 0$ is satisfied.

All other two and three dimensional subalgebras listed in Table 6.2 do not give meaningful reductions of the original equation (6.2), so we will not regard them in detail.

6.2.4 System of optimal subalgebras of L_4^{HARA}. Case $\kappa = r\gamma$, i.e. $L_4^{HARA} = A_{3,5}^{\gamma} \bigoplus A_1$

In Section 6.2.2 we worked with the case $\kappa \neq r\gamma$ now we study the special case $\kappa = r\gamma$. When $\kappa = r\gamma$ as we have mentioned above L_4^{HARA} has a different structure according to the classification [50]. We substitute now $\kappa = r\gamma$ into (6.11) and regard the following equation

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl + \delta h)V_l(t,l,h) + (\mu - \delta)hV_h(t,l,h) \\ - & \frac{(\alpha - r)^2 V_l^2(t,l,h) + 2(\alpha - r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ + & \frac{(1-\gamma)^2}{\gamma} e^{-\frac{r\gamma t}{1-\gamma}} V_l(t,l,h)^{-\frac{\gamma}{1-\gamma}} - \frac{1-\gamma}{\gamma} e^{-r\gamma t} = 0, \quad V(l,h,t) \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{6.55}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function $V(t,l,h)$

$$\pi(t,l,h) = -\frac{\eta\rho\sigma h V_{lh}(t,l,h) + (\alpha - r)V_l(t,l,h)}{\sigma^2 V_{ll}(t,l,h)}, \tag{6.56}$$

$$c(t,l,h) = (1-\gamma) V_l^{-\frac{1}{1-\gamma}} e^{-\frac{r\gamma}{1-\gamma}t}. \tag{6.57}$$

In this case we transform the old basis of $L_4^{HARA} = < \mathbf{U}_1, \mathbf{U}_2, \mathbf{U}_3, \mathbf{U}_4 >$ described in (5.23) and (5.24) into the following one

$$e_1 = \mathbf{U}_2,\ e_2 = \mathbf{U}_1,\ e_3 = -\frac{1}{r}\mathbf{U}_4,\ e_4 = \mathbf{U}_3 + \frac{1}{r}\mathbf{U}_4,$$

where we use the relation $\kappa = r\gamma$. Now the new basis looks like this

$$\begin{aligned} e_1 &= e^{rt}\frac{\partial}{\partial l}, \\ e_2 &= \frac{\partial}{\partial V}, \\ e_3 &= -\frac{1}{r}\frac{\partial}{\partial t} + \gamma V \frac{\partial}{\partial V}, \\ e_4 &= l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \frac{1}{r}\frac{\partial}{\partial t} + \frac{1-\gamma}{r\gamma}e^{-r\gamma t}\frac{\partial}{\partial V}. \end{aligned} \tag{6.58}$$

In this basis algebra L_4^{HARA} has only two non-zero commutation relations

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = \gamma e_2. \tag{6.59}$$

Now we can see that L_4^{HARA} corresponds to the algebra of the type $A_{3,5}^{\gamma} \bigoplus A_1$, in the notation of [50]. The system of optimal subalgebras for this algebra is listed in Table 6.3.

Dimension of the subalgebra	System of optimal subalgebras of algebra L_4^{HARA}, $\kappa = r\gamma$
1	$h_1 = \langle e_1 \rangle\,,\ h_2 = \langle e_2 \rangle\,,\ h_3 = \langle e_4 \rangle\,,\ h_4 = \langle\ e_1 \pm e_4 \rangle\,,\ h_5 = \langle\ e_2 \pm e_4 \rangle\,,$ $h_6 = \langle\ e_3 + \omega e_4 \rangle\,,\ h_7 = \langle\ e_1 \pm e_2 + \omega e_4 \rangle$
2	$h_8 = \langle e_1, e_2 \rangle\,,\ h_9 = \langle e_1, e_4 \rangle\,,\ h_{10} = \langle e_2, e_4 \rangle\,,\ h_{11} = \langle e_3, e_4 \rangle\,,$ $h_{12} = \langle e_1 \pm e_2, e_4 \rangle\,,\ h_{13} = \langle e_1, e_2 \pm e_4 \rangle\,,\ h_{14} = \langle e_1 \pm e_4, e_2 + \omega e_4 \rangle\,,$ $h_{15} = \langle e_3 + \omega e_4, e_1 \rangle\,,\ h_{16} = \langle e_3 + \omega e_4, e_2 \rangle$
3	$h_{17} = \langle e_1, e_2, e_4 \rangle\,, h_{18} = \langle e_3, e_4, e_1 \rangle\,,$ $h_{19} = \langle e_3, e_4, e_2 \rangle\,, h_{20} = \langle e_3 + \omega e_4, e_1, e_2 \rangle$

Table 6.3: [50] The optimal system of one, two and three dimensional subalgebras of L_4^{HARA}, if $\kappa = r\gamma$, where ω is a parameter such that $-\infty < \omega < \infty$.

6.2.5 One-dimensional subalgebras of L_4^{HARA}, $\kappa = r\gamma$, and corresponding reductions

We use now the Table 6.3 to list all non equivalent possible reductions of the three dimensional PDE (6.55) in the the case $\kappa = r\gamma$. As we have shown before the first two subalgebras h_1 and h_2, spanned by $h_1 = \left\langle e^{rt}\frac{\partial}{\partial l}\right\rangle$ and $h_2 = \left\langle \frac{\partial}{\partial W}\right\rangle$ correspondingly, provide no meaningful reductions of (6.11) or, correspondingly, (6.55). Let us move on to the next case.

Case $H_3(h_3)$. This subalgebra is spanned by e_4, i.e.

$$h_3 =< e_4 >= \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \frac{1}{r}\frac{\partial}{\partial t} + \frac{1-\gamma}{r\gamma}e^{-r\gamma t}\frac{\partial}{\partial V}\right\rangle.$$

Solving the corresponding characteristic system we find the following invariants of the subgroup H_3

$$\begin{aligned} inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = rt - \log h, && (6.60) \\ inv_3 &= W(z,\tau) = V + \frac{1-\gamma}{r\gamma^2} e^{-r\gamma t}. && (6.61) \end{aligned}$$

Substituting these invariants into (6.55) we obtain a two dimensional PDE which describes all solutions that are invariant under the action of the subgroup H_3

$$\begin{aligned} & rW_\tau(z,\tau) + \frac{1}{2}\eta^2 \left(2zW_z(z,\tau) + z^2 W_{zz}(z,\tau)\right) \\ + \ & (rz+\delta)W_z(z,\tau) - (\mu-\delta)zW_z(z,\tau) \\ - \ & \frac{(\alpha-r)^2 W_z^2(z,\tau) - 2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau) + zW_{zz}(z,\tau))}{2\sigma^2 W_{zz}(z,\tau)} \qquad (6.62) \\ - \ & \frac{\eta^2\rho^2\sigma^2(W_z(z,\tau) + zW_{zz}(z,\tau))^2}{2\sigma^2 W_{zz}(z,\tau)} + \frac{(1-\gamma)^2}{\gamma} e^{-\frac{\gamma}{1-\gamma}\tau} W_z^{-\frac{\gamma}{1-\gamma}} = 0, \\ & W(z,\tau) \underset{\tau\to\infty}{\longrightarrow} 0. \end{aligned}$$

Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the value function $W(z,\tau)$

$$(6.63)$$

$$\begin{aligned} \pi(z,\tau,h) &= h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma - \alpha + r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), && (6.64) \\ c(z,\tau,h) &= h(1-\gamma)W_z(z,\tau)^{-\frac{1}{1-\gamma}} e^{-\frac{\gamma}{1-\gamma}\tau}. && (6.65) \end{aligned}$$

Case $H_4(h_4)$. As one can see in Table 6.3 this subalgebra is spanned by

$$h_4 = < e_1 \pm e_4 > = \left\langle (l \pm e^{rt})\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \frac{1}{r}\frac{\partial}{\partial t} + \frac{1-\gamma}{r\gamma}e^{-r\gamma t}\frac{\partial}{\partial V}\right\rangle.$$

We find the following invariants solving the characteristic system of equations

$$\begin{aligned} inv_1 &= x = le^{-rt} \mp rt, \quad inv_2 = y = he^{-rt}, && (6.66) \\ inv_3 &= W(x,y) = V + \frac{1-\gamma}{r\gamma^2}e^{-r\gamma t}. && (6.67) \end{aligned}$$

We use the invariants z and τ (6.66) of H_4 as independent variables and the invariant $W(z,\tau)$ (6.67) as the dependent variable. This way we reduce equation

(6.55) to a two dimensional PDE that looks as follows

$$
\begin{aligned}
\mp\ & rW_x(x,y) + \delta y W_x(x,y) + (\mu-\delta) y W_y(x,y) + \frac{1}{2}\eta^2 y^2 W_{yy}(x,y) \\
-\ & \frac{(\alpha-r)^2 W_x^2(x,y) - 2(\alpha-r)\eta\rho y W_x(x,y) W_{xy}(x,y) + \eta^2\rho^2\sigma^2 y^2 W_{xy}^2(x,y)}{2\sigma^2 W_{xx}(x,y)} \\
+\ & \frac{(1-\gamma)^2}{\gamma} W_x^{-\frac{\gamma}{1-\gamma}}(x,y) = 0, \quad W(x,0) \underset{x\to\mp\infty}{\longrightarrow} 0. \qquad (6.68)
\end{aligned}
$$

Here the investment $\pi(x,y,h)$ and consumption $c(x,y,h)$ look as follows in terms of the value function $W(x,y)$

$$
\begin{aligned}
\pi(x,y,h) &= -h\frac{\eta\rho\sigma W_{xy}(x,y) + (\alpha-r)y^{-1}W_x(x,y)}{\sigma^2 W_{xx}(x,y)}, && (6.69) \\
c(x,y,h) &= h(1-\gamma)y^{-1}W_x^{-\frac{1}{1-\gamma}}(x,y). && (6.70)
\end{aligned}
$$

Case $H_5(h_5)$. This subalgebra h_5 is spanned by

$$
h_5 =< e_2 \pm e_4 >= \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \frac{1}{r}\frac{\partial}{\partial t} + \left(\pm 1 + \frac{1-\gamma}{r\gamma}e^{-r\gamma t}\right)\frac{\partial}{\partial V}\right\rangle.
$$

We find the following invariants of the subgroup H_5

$$
\begin{aligned}
inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = rt - \log h, && (6.71) \\
inv_3 &= W(z,\tau) = V \mp rt + \frac{1-\gamma}{r\gamma^2}e^{-r\gamma t}. && (6.72)
\end{aligned}
$$

Substituting the invariants of H_5 (6.71) as independent variables z,τ and the invariant (6.72) as the dependent variable $W(z,\tau)$ into (6.55) we obtain a two dimensional PDE

$$
\begin{aligned}
& rW_\tau(z,\tau) \pm r + \frac{1}{2}\eta^2\left(2zW_z(z,\tau) + z^2 W_{zz}(z,\tau)\right) \\
+\ & (rz+\delta)W_z(z,\tau) - (\mu-\delta) z W_z(z,\tau) \\
-\ & \frac{(\alpha-r)^2 W_z^2(z,\tau) - 2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau) + zW_{zz}(z,\tau))}{2\sigma^2 W_{zz}(z,\tau)} \qquad (6.73) \\
-\ & \frac{\eta^2\rho^2\sigma^2(W_z(z,\tau)+zW_{zz}(z,\tau))^2}{2\sigma^2 W_{zz}(z,\tau)} + \frac{(1-\gamma)^2}{\gamma}e^{-\frac{\gamma}{1-\gamma}\tau}W_z^{-\frac{\gamma}{1-\gamma}} = 0, \\
& W(z,\tau) \underset{\tau\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the value function $W(z,\tau)$

$$\begin{aligned}\pi(z,\tau,h) &= h\left(\frac{\eta\rho}{\sigma}z+\frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), &(6.74)\\ c(z,\tau,h) &= h(1-\gamma)W_z(z,\tau)^{-\frac{1}{1-\gamma}}e^{-\frac{\gamma}{1-\gamma}\tau}. &(6.75)\end{aligned}$$

This equation for the new dependent variable $W(z,\tau)$ differs from (6.62) only by the term r as well as the corresponding invariant (6.72) differs from (6.61) by the linear term rt.

Case $H_6(h_6)$. This subalgebra is spanned by

$$h_6 =< e_3+\omega e_4 >= \left\langle \omega l\frac{\partial}{\partial l}+\omega h\frac{\partial}{\partial h}+\frac{\omega-1}{r}\frac{\partial}{\partial t}+\left(\gamma V+\omega\frac{1-\gamma}{r\gamma}e^{-r\gamma t}\right)\frac{\partial}{\partial V}\right\rangle.$$

If $\omega=1$ that case is equivalent to (6.27) that we have regarded earlier, the only difference is that we need to take into consideration that now $\kappa=r\gamma$. For all the values of $\omega\neq 1$ we find the following invariants of the subgroup H_6 solving the corresponding characteristic system

$$\begin{aligned}inv_1 &= z=\frac{l}{h},\quad inv_2=\tau=\frac{r\omega}{\omega-1}t-\log h, &(6.76)\\ inv_3 &= W(z,\tau)=Ve^{-\frac{r\gamma}{\omega-1}t}+\frac{1-\gamma}{r\gamma^2}e^{-\frac{\omega r\gamma}{\omega-1}t},\quad \omega\neq 1. &(6.77)\end{aligned}$$

We use the invariants of H_6 (6.76) as independent variables z,τ and the invariant (6.77) as the dependent variable $W(z,\tau)$ in (6.55) and obtain a two dimensional equation

$$\begin{aligned}&\frac{r\gamma}{\omega-1}W(z,\tau)+\frac{r\omega}{\omega-1}W_\tau(z,\tau)+\frac{1}{2}\eta^2\left(2zW_z(z,\tau)+z^2W_{zz}(z,\tau)\right)\\ +\;&(rz+\delta)W_z(z,\tau)-(\mu-\delta)zW_z(z,\tau) \qquad (6.78)\\ -\;&\frac{(\alpha-r)^2W_z^2(z,\tau)-2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau)+zW_{zz}(z,\tau))}{2\sigma^2W_{zz}(z,\tau)}\\ +\;&\frac{\eta^2\rho^2\sigma^2(W_z(z,\tau)+zW_{zz}(z,\tau))^2}{2\sigma^2W_{zz}(z,\tau)}+\frac{(1-\gamma)^2}{\gamma}e^{-\frac{\gamma}{1-\gamma}\tau}W_z^{-\frac{\gamma}{1-\gamma}}(z,\tau)=0,\\ &W(z,\tau)\underset{\tau\to\infty}{\longrightarrow}0.\end{aligned}$$

Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the value function $W(z,\tau)$

$$\begin{aligned}
\pi(z,\tau,h) &= h\left(\frac{\eta\rho}{\sigma}z+\frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), && (6.79)\\
c(z,\tau,h) &= h(1-\gamma)e^{\frac{-\gamma}{1-\gamma}\tau}W_z^{-\frac{1}{1-\gamma}}(z,\tau). && (6.80)
\end{aligned}$$

Case $H_7(h_7)$. This subalgebra is spanned by

$$\begin{aligned}
h_7 &= < e_1 \pm e_2 + \omega e_4 >= \\
&\left\langle \left(e^{rt}+\omega l\right)\frac{\partial}{\partial l}+\omega h\frac{\partial}{\partial h}+\frac{\omega}{r}\frac{\partial}{\partial t}+\left(\pm V+\omega\frac{1-\gamma}{r\gamma}e^{-r\gamma t}\right)\frac{\partial}{\partial V}\right\rangle.
\end{aligned}$$

Let us note here that if $\omega=0$ this case is equivalent to the case (6.10) under a condition $\kappa=r\gamma$ that we have regarded before. If $\omega\neq 0$ we can find the following invariants of the subgroup H_7

$$\begin{aligned}
inv_1 &= x=le^{-rt}-\frac{r}{\omega}t, \quad inv_2=y=he^{-rt}, && (6.81)\\
inv_3 &= W(x,y)=V\mp\frac{r}{\omega}t+\frac{1-\gamma}{r\gamma^2}e^{-r\gamma t}, \quad \omega\neq 0. && (6.82)
\end{aligned}$$

Substituting the invariants of H_7 (6.81) as the independent variables x and y and the invariant (6.82) as the dependent variable $W(x,y)$ into (6.55) we obtain a two dimensional PDE

$$\begin{aligned}
&\pm \frac{r}{\omega}-\frac{r}{\omega}W_x(x,y)+\frac{1}{2}\eta^2y^2W_{yy}(x,y)+\delta yW_x(x,y)+(\mu-\delta)yW_y(z,y) \quad (6.83)\\
&- \frac{(\alpha-r)^2W_x^2(x,y)-2(\alpha-r)\eta\rho yW_x(x,y)W_{xy}(x,y)+\eta^2\rho^2\sigma^2y^2W_{xy}^2(x,y)}{2\sigma^2W_{xx}(x,y)}\\
&+ \frac{(1-\gamma)^2}{\gamma}W_x^{-\frac{\gamma}{1-\gamma}}(x,y)=0, \quad W(x,0)\underset{x\to\mp\infty}{\longrightarrow}0.
\end{aligned}$$

Here the investment $\pi(x,y,h)$ and consumption $c(x,y,h)$ look as follows in terms of the value function $W(x,y)$

$$\begin{aligned}
\pi(x,y,h) &= -h\frac{\eta\rho\sigma W_{xy}(x,y)+(\alpha-r)y^{-1}W_x(x,y)}{\sigma^2W_{xx}(x,y)}, && (6.84)\\
c(x,y,h) &= h(1-\gamma)y^{-1}W_x^{-\frac{1}{1-\gamma}}(x,y). && (6.85)
\end{aligned}$$

6.2.6 Two-dimensional subalgebras of L_4^{HARA}, $\kappa = r\gamma$, and corresponding reductions

In this section we continue working with Table 6.3 and look for the deeper reductions that can reduce PDE (6.55) to an ordinary differential equation. First of all, let us note that all two dimensional subalgebras listed in Table 6.2 but for h_{11} do not give any meaningful reduction of the original equation (6.55), so we will not regard them in detail.

Case $H_{11}(h_{11})$. The first two dimensional sub algebra listed in Table 6.2 is subalgebra $h_{11} =< e_3, e_4 >$ spanned by two generators defined as follows

$$\begin{aligned} e_3 &= -\frac{1}{r}\frac{\partial}{\partial t} + \gamma V\frac{\partial}{\partial V}, \\ e_4 &= l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \frac{1}{r}\frac{\partial}{\partial t} + \frac{1-\gamma}{r\gamma}e^{-r\gamma t}\frac{\partial}{\partial V}. \end{aligned} \tag{6.86}$$

As earlier, we look for a solution that would satisfy both characteristic systems. Since we have already looked on one of the generators, we can use our previous knowledge and reformulate one of the equations of the characteristic system in terms of invariant variables of the other one. We have found a general form of the invariants of e_4 in (6.60), (6.61). Let us list them here again for the convenience of the reader

$$\begin{aligned} inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = rt - \log h, \\ inv_3 &= W(z,\tau) = V + \frac{1-\gamma}{r\gamma^2}e^{-r\gamma t}. \end{aligned}$$

If we rewrite e_3 (6.86) in terms of these three invariants z, τ and W as new independent and dependent variables correspondingly, we obtain

$$e_3 = -\frac{\partial}{\partial \tau} + \gamma W\frac{\partial}{\partial W}. \tag{6.87}$$

Solving a corresponding characteristic system $\frac{d\tau}{-1} = \frac{dW}{\gamma W}$ we obtain a new invariant

$$inv_{e_3} = Y(z) = W(z,\tau)e^{\gamma\tau}. \tag{6.88}$$

This way we obtain an invariant solution under the action of two parameter subgroup H_{11}. It means that we can take $Y(z)$ as a new dependent variable in

(6.62) and z as a new independent one. Substituting these invariants into PDE (6.62) we obtain a new ODE

$$\begin{aligned} &- r\gamma Y(z) + \frac{1}{2}\eta^2 \left(2zY_z(z) + z^2 Y_{zz}(z)\right) + (rz+\delta)Y_z(z) - (\mu-\delta)zY_z(z) \\ &- \frac{(\alpha-r)^2 Y_z^2(z) - 2(\alpha-r)\eta\rho Y_z(z)(Y_z(z)+zY_{zz}(z)) + \eta^2\rho^2\sigma^2(Y_z(z)-zY_{zz}(z))^2}{2\sigma^2 Y_{zz}(z)} \\ &+ \frac{(1-\gamma)^2}{\gamma} Y_z^{-\frac{\gamma}{1-\gamma}}(z) = 0. \end{aligned} \tag{6.89}$$

Naturally, the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ in terms of $Y(z)$ look now as follows

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{Y_z(z)}{Y_{zz}(z)}\right), \tag{6.90}$$

$$c(z,\tau,h) = h(1-\gamma)Y_z(z)^{-\frac{1}{1-\gamma}}. \tag{6.91}$$

In terms of original variables l, h, t and V the substitution looks as follows

$$\begin{aligned} z &= \frac{l}{h}, \quad \tau = rt - \log h, \\ Y(z) &= \left(V(t,l,h)e^{\gamma rt} + \frac{1-\gamma}{r\gamma^2}\right)h^{-\gamma}. \end{aligned}$$

This substitution is equivalent to the substitution (7.43) under the condition $\kappa = r\gamma$. It is also important to note that if we obtain a solution $Y(z)$ for (6.89) we obtain the value function that in terms of original variables looks like

$$V(t,l,h) = e^{-\kappa t}h^{\gamma}Y(l/h) - \frac{1-\gamma}{\gamma\kappa}e^{-\kappa t},$$

and the condition $V(l,h,t) \underset{t\to\infty}{\longrightarrow} 0$ is satisfied.

6.3 Results of the chapter

In this chapter we have found all Lie type reductions of the PDE with HARA utility function and different liquidation time distributions that follow from the results of Lie group analysis of the problem that was carried out in Chapter 5.

Using the notation provided in [50] we show that for a general liquidation time distribution L_3^{HARA} can be classified as $A_{3,5}^{\gamma}$, whereas in a special case of exponentially distributed liquidation function the admitted Lie algebra is four dimensional is either classified as $A_2 \oplus A_2$ in terms of the notation used in [50] or, if $\kappa = r\gamma$, i.e. there is a certain dependency between the parameter of HARA utility function and the corresponding liquidation time distribution, the symmetry algebra L_4^{HARA} has the structure that corresponds to $A_{3,5}^{\gamma} \bigoplus A_1$.

Using a system of optimal subalgebras for all admitted algebras allows us to provide all non equivalent reductions of the studied equations and describe the solutions which can not be transformed to each other with a help of the transformations from the admitted symmetry group. For every case we list all possible Lie type reductions of the problem. The reduced equations that are two dimensional PDEs or in some special cases are even ODEs. Such equations are more convenient for further analytical or numerical studies.

We also show how one can rewrite corresponding optimal policies in every case. They can be described using solutions of the reduced equations. One can see that optimal policies tend to classical Merton policies as $h \to 0$, that is only logical, since by construction this situation should correspond to a portfolio without illiquid asset.

In there next chapter we carry out the same analysis for the case of logarithmic utility.

7

Reductions of PDEs with logarithmic utility

Now let us discuss the reductions of the equations with logarithmic utility function. These reductions follow from the results of Lie group analysis of the problem that was carried out in Chapter 5. We obtain the complete set of possible symmetry reductions of three dimensional PDE (5.35) for the case of log-utility function and a general liquidation time distribution. We also regard a special case of an exponentially distributed liquidation time in Section 7.2. The logarithmic utility could be regarded as a special case of HARA-utility, as we have mentioned earlier (5.19), moreover we see that $L_{3,4}^{HARA} \underset{\gamma \to 0}{\longrightarrow} L_{3,4}^{LOG}$. Yet logarithmic utility is widely used in financial mathematics and therefore deserves our attention.

7.1 Reductions in the case of general liquidation time distribution and logarithmic utility

Analogously to the previous sections we look for the optimal system of subalgebras for the Lie algebra L_3^{LOG} (5.38) admitted by the equation (5.35). In order to describe all non equivalent symmetry reductions and not loose any of them one has to study an optimal system of Lie subalgebras. We present here for a convenience of the reader the main equation for this chapter

$$\begin{aligned} & V_t(t,l,h) + \frac{1}{2}\eta^2 h^2 V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\ - & \frac{(\alpha-r)^2 V_l^2(t,l,h) + 2(\alpha-r)\eta\rho h V_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2 h^2 {V_{lh}}^2(t,l,h)}{2\sigma^2 V_{ll}(t,l,h)} \\ - & \overline{\Phi}(t)\left(\log V_l(t,l,h) - \log\overline{\Phi}(t) + 1\right) = 0, \quad V \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{7.1}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function $V(t,l,h)$

$$\pi(t,l,h) = -\frac{\eta\rho\sigma h V_{lh}(t,l,h) + (\alpha-r)V_l(t,l,h)}{\sigma^2 V_{ll}(t,l,h)}, \tag{7.2}$$

$$c(t,l,h) = \frac{\overline{\Phi}(t)}{V_l(t,l,h)}. \tag{7.3}$$

We list all symmetry reductions that can reduce the dimension of the problem by one or two and study them closely in the next Section. We demonstrate how the problem (7.1) transforms after every substitution.

7.1.1 System of optimal subalgebras of L_3^{LOG}

At first let us reassign the basis of L_3^{LOG} (5.38) so that it is possible to use the classification [50] in a convenient way. This can be done in the following way $\mathbf{U}_1 = -e_3, \mathbf{U}_2 = e_2, \mathbf{U}_3 = -e_1$. The basis is now defined as

$$e_1 = -l\frac{\partial}{\partial l} - h\frac{\partial}{\partial h} + \int \overline{\Phi}(t)dt\frac{\partial}{\partial V}, \quad e_2 = e^{rt}\frac{\partial}{\partial l}, \quad e_3 = -\frac{\partial}{\partial V}. \tag{7.4}$$

In the new basis the algebra has just one non zero commutation relation

$$[e_1, e_2] = e_2. \tag{7.5}$$

Now we can see that L_3^{LOG} corresponds to $A_1 \bigoplus A_2$ case, in the notation of [50]. The system of optimal subalgebras is given in Table 7.1.

7.1.2 One dimensional subalgebras of L_3^{LOG} and corresponding reductions

Let us now look at one dimensional subalgebras of L_3^{LOG} one by one to find all non equivalent reductions of (7.1).

Dimension of the subalgebra	System of optimal subalgebras of algebra L_3^{LOG}
1	$h_1 = \langle e_1 \cos\beta + e_3 \sin\beta \rangle\,,\ h_2 = \langle e_2 \pm e_3 \rangle\,,\ h_3 = \langle e_2 \rangle$
2	$h_4 = \langle e_1, e_3 \rangle\,,\ h_5 = \langle e_2, e_3 \rangle\,,\ h_6 = \langle e_1 + \omega e_3, e_2 \rangle$

Table 7.1: [50] The optimal system of one and two dimensional subalgebras of L_3^{LOG} (5.23), where ω and β are parameters such that $-\infty < \omega < \infty$ and $0 \leq \beta < \pi$.

Case $H_1(h_1)$. This subalgebra is spanned by the generator

$$\begin{aligned} h_1 &= < e_1 \cos\beta + e_3 \sin\beta > \\ &= \left\langle \cos\beta l \frac{\partial}{\partial l} + \cos\beta h \frac{\partial}{\partial h} + \left(\sin\beta - \cos\beta \int \overline{\Phi}(t) dt \right) \frac{\partial}{\partial V} \right\rangle . \end{aligned}$$

Solving a corresponding characteristic system for the invariants of H_1 we obtain independent invariants

$$inv_1 = t, \quad inv_2 = z = \frac{l}{h} \tag{7.6}$$

$$inv_3 = W(z,t) = V + \log h \left(\tan\beta + \int \overline{\Phi}(t) dt \right). \tag{7.7}$$

Substituting (7.6) as new independent variables t, z and (7.7) as a new dependent variable $W(z,t)$ into (5.35) we get

$$\begin{aligned} & W_t(z,t) + \frac{1}{2}\eta^2 \left(2zW_z(z,t) + z^2 W_{zz}(z,t) \right) \\ + & (rz+\delta) W_z(z,t) - (\mu - \delta) z W_z(z,t) \\ - & \frac{(\alpha - r)^2 W_z^2(z,t) - 2(\alpha - r)\eta\rho W_z(z,t)(W_z(z,t) + zW_{zz}(z,t))}{2\sigma^2 W_{zz}(z,t)} \\ - & \frac{\eta^2 \rho^2 \sigma^2 (W_z(z,t) - zW_{zz}(z,t))^2}{2\sigma^2 W_{zz}(z,t)} \\ - & \overline{\Phi}(t) \log W_z(z,t) - \left(\frac{1}{2}\eta^2 - \mu + \delta \right) \int \overline{\Phi}(t) dt + \overline{\Phi}(t)(\log \overline{\Phi}(t) - 1) \\ - & \left(\frac{1}{2}\eta^2 - \mu + \delta \right) \tan\beta = 0. \end{aligned} \tag{7.8}$$

Indeed the condition $V \underset{t\to\infty}{\longrightarrow} 0$ holds only for the situation when $\beta = 0$. When $\beta = 0$, the investment $\pi(z,t,h)$ and consumption $c(z,t,h)$ look as follows in terms of the value function W

$$\pi(z,t,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,t)}{W_{zz}(z,t)}\right), \tag{7.9}$$

$$c(z,t,h) = h\frac{\overline{\Phi}(t)}{W_z(z,t)}. \tag{7.10}$$

Cases $H_2(h_1)$ defined by subalgebra $h_2 = \left\langle e^{rt}\frac{\partial}{\partial l}\right\rangle$ and $H_3(h_3)$ defined by $h_3 = \left\langle \frac{\partial}{\partial V}\right\rangle$ were in principle discussed in Section 6.1.2 hence we do not repeat this discussion, we just emphasize that they do not provide any reductions. We can see that for a general liquidation time distribution we have only one meaningful Lie type reduction of the equation (7.1). This does not mean that the problem can not be simplified or solved using other methods, nevertheless the two dimensional such as (7.8) is usualluy more convenient to work with.

7.2 A special case of an exponential liquidation time distribution and log utility function

In Theorem 13 we have shown that the case of an exponential liquidation time distribution is a special one and the equation (5.35) admits an extended Lie algebra. There are four Lie symmetries in this case, describe in (5.38) and (5.39). We would like to pay some special attention to the case of a logarithmic utility since this particular case is broadly regarded in the literature. The equation (7.1) we study in this section now, when we insert $\overline{\Phi}(t) = e^{-\kappa t}$, looks as follows

$$\begin{aligned}
& V_t(t,l,h) + \frac{1}{2}\eta^2h^2V_{hh}(t,l,h) + (rl+\delta h)V_l(t,l,h) + (\mu-\delta)hV_h(t,l,h) \\
- & \frac{(\alpha-r)^2V_l^2(t,l,h) + 2(\alpha-r)\eta\rho hV_l(t,l,h)V_{lh}(t,l,h) + \eta^2\rho^2\sigma^2h^2{V_{lh}}^2(t,l,h)}{2\sigma^2V_{ll}(t,l,h)} \\
- & e^{-\kappa t}\left(\log V_l(t,l,h) + \kappa t + 1\right) = 0, \quad V \underset{t\to\infty}{\longrightarrow} 0.
\end{aligned} \tag{7.11}$$

Here the investment $\pi(t,l,h)$ and consumption $c(t,l,h)$ look as follows in terms of the value function $V(t,l,h)$

$$\pi(t,l,h) = -\frac{\eta\rho\sigma hV_{lh}(t,l,h) + (\alpha-r)V_l(t,l,h)}{\sigma^2V_{ll}(t,l,h)}, \tag{7.12}$$

$$c(t,l,h) = \frac{e^{-\kappa t}}{V_l(t,l,h)}. \tag{7.13}$$

7.2.1 System of optimal subalgebras of L_4^{LOG}

As in previous cases we change the basis of L_4^{LOG} to use the convenient classification provided in [50]. Let us at first transform the basis as follows

$$e_1 = \frac{r\mathbf{U}_3 + \mathbf{U}_4}{\kappa}, \quad e_2 = \mathbf{U}_1, \quad e_3 = -\mathbf{U}_3, \quad e_4 = \mathbf{U}_2.$$

Now the generators of the new basis of $L_4 =< e_1, e_2, e_3, e_4 >$ look like this

$$\begin{aligned} e_1 &= \frac{r}{\kappa} l \frac{\partial}{\partial l} + \frac{r}{\kappa} h \frac{\partial}{\partial h} + \frac{1}{\kappa}\frac{\partial}{\partial t} - \left(V - \frac{r}{\kappa^2} e^{-\kappa t}\right)\frac{\partial}{\partial V}, \\ e_2 &= \frac{\partial}{\partial V}, \\ e_3 &= -l\frac{\partial}{\partial l} - h\frac{\partial}{\partial h} - \frac{1}{\kappa} e^{-\kappa t}\frac{\partial}{\partial V}, \\ e_4 &= e^{rt}\frac{\partial}{\partial l}. \end{aligned}$$

In this basis there are only two non-zero commutation relations on L_4^{LOG}

$$[e_1, e_2] = e_2, \quad [e_3, e_4] = e_4. \tag{7.14}$$

We see that L_4^{LOG} corresponds to $2A_2$, in the notation of [50]. The system of optimal subalgebras of L_4^{LOG} is listed in Table 7.2.

Dimension of the subalgebra	System of optimal subalgebras of algebra L_4^{LOG}
1	$h_1 = \langle e_2 \rangle$, $h_2 = \langle e_3 \rangle$, $h_3 = \langle e_4 \rangle$, $h_4 = \langle e_1 + \omega e_3 \rangle$, $h_5 = \langle e_1 \pm e_4 \rangle$, $h_6 = \langle e_2 \pm e_4 \rangle$, $h_7 = \langle e_2 \pm e_3 \rangle$
2	$h_8 = \langle e_1, e_3 \rangle$, $h_9 = \langle e_1, e_4 \rangle$, $h_{10} = \langle e_2, e_3 \rangle$, $h_{11} = \langle e_2, e_4 \rangle$, $h_{12} = \langle e_1 + \omega e_3, e_2 \rangle$, $h_{13} = \langle e_3 + \omega e_1, e_4 \rangle$, $h_{14} = \langle e_1 \pm e_4, e_2 \rangle$, $h_{15} = \langle e_3 \pm e_2, e_4 \rangle$, $h_{16} = \langle e_1 + e_3, e_2 \pm e_4 \rangle$
3	$h_{17} = \langle e_1, e_3, e_2 \rangle$, $h_{18} = \langle e_1, e_4, e_2 \rangle$, $h_{19} = \langle e_1, e_3, e_4 \rangle$, $h_{20} = \langle e_2, e_3, e_4 \rangle$, $h_{21} = \langle e_1 \pm e_3, e_2, e_4 \rangle$, $h_{22} = \langle e_1 + \omega e_3, e_2, e_4 \rangle$

Table 7.2: [50] The optimal system of one, two and three dimensional subalgebras of L_4^{LOG}, where ω is a parameter such that $-\infty < \omega < \infty$.

7.2.2 One dimensional subalgebras of L_4^{LOG} and corresponding reductions

Now we are going to study all invariant reductions of the problem (7.11). Let us first note that the subalgebras h_1, h_3 and h_6, defined as $h_1 = \langle \frac{\partial}{\partial V} \rangle$, $h_3 = \langle e^{rt} \frac{\partial}{\partial l} \rangle$ and $h_6 = \langle \frac{\partial}{\partial V} \pm e^{rt} \frac{\partial}{\partial l} \rangle$ correspondingly, do not give us any interesting reductions so we omit the detailed study of these cases here. We start with a first interesting and non-trivial case.

Case $H_2(h_2)$. The sub algebra is spanned by the generator e_3, i.e.

$$h_2 =< e_3 >= \left\langle -l\frac{\partial}{\partial l} - h\frac{\partial}{\partial h} - \frac{1}{\kappa}e^{-\kappa t}\frac{\partial}{\partial V} \right\rangle .$$

To find all invariants of the subgroup H_2 we solve the corresponding characteristic system of equations and obtain

$$\begin{aligned} inv_1 &= z = \frac{l}{h}, \quad inv_2 = t, \\ inv_3 &= W(z,t) = \kappa e^{\kappa t} V - \log h. \end{aligned} \tag{7.15}$$

Substituting two invariants of H_2 (7.15) as the new independent variables z, t and (7.15) as the dependent variable $W(z,t)$ into (7.11) we get

$$\begin{aligned} W_t(z,t) &- \kappa W(z,t) + \frac{1}{2}\eta^2 \left(2zW_z(z,t) + z^2 W_{zz}(z,t)\right) \\ &+ (rz+\delta)W_z(z,t) - (\mu-\delta)zW_z(z,t) \\ &- \frac{(\alpha-r)^2 W_z^2(z,t) - 2(\alpha-r)\eta\rho W_z(z,t)(W_z(z,t) + zW_{zz}(z,t))}{2\sigma^2 W_{zz}(z,t)} \\ &- \frac{\eta^2\rho^2\sigma^2(W_z(z,t)+zW_{zz}(z,t))^2}{2\sigma^2 W_{zz}(z,t)} - \kappa \log W_z(z,t) \\ &- \left(\frac{1}{2}\eta^2 - \mu + \delta\right) + \kappa(\log\kappa - 1) = 0, \quad W \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{7.16}$$

Here the investment $\pi(z,t,h)$ and consumption $c(z,t,h)$ look as follows in terms of the function $W(z,t)$

$$\pi(z,t,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma - \alpha + r}{\sigma^2}\frac{W_z(z,t)}{W_{zz}(z,t)}\right), \tag{7.17}$$

$$c(z,t,h) = h\frac{\kappa}{W_z(z,t)}. \tag{7.18}$$

Case $H_4(h_4)$. The subalgebra that corresponds to H_4 is spanned by $e_1 + \omega e_3$ and defined as

$$\begin{aligned} h_4 &= < e_1 + \omega e_3 > \\ &= \left\langle \left(\frac{r}{\kappa} - \omega\right) l \frac{\partial}{\partial l} + \left(\frac{r}{\kappa} - \omega\right) h \frac{\partial}{\partial h} + \frac{1}{\kappa}\frac{\partial}{\partial t} - \left(V - \frac{r - \omega\kappa}{\kappa^2} e^{-\kappa t}\right) \frac{\partial}{\partial V} \right\rangle. \end{aligned}$$

We need to regard two special cases $\omega = r/\kappa$ and $\omega \neq r/\kappa$ here.

If $\omega = r/\kappa$ this case h_4 is defined as

$$h_4 =< e_1 + \frac{r}{\kappa} e_3 >= \left\langle \frac{1}{\kappa}\frac{\partial}{\partial t} - V\frac{\partial}{\partial V} \right\rangle, \quad \omega = \frac{r}{\kappa}.$$

The invariants for this case are as follows

$$inv_1 = l, \quad inv_2 = h, \tag{7.19}$$

$$inv_3 = W(l,h) = Ve^{\kappa t} \tag{7.20}$$

Using two invariants (7.19) as the new independent variables and (7.20) as the dependent variable in (7.11) we obtain a two dimensional PDE

$$\begin{aligned} &- \kappa W(l,h) + \frac{1}{2}\eta^2 h^2 W_{hh}(l,h) + (rl + \delta h) W_l(l,h) + (\mu - \delta) h W_h(l,h) \\ &- \frac{(\alpha - r)^2 W_l^2(l,h) + 2(\alpha - r)\eta\rho h W_l(l,h) W_{lh}(l,h) + \eta^2\rho^2\sigma^2 h^2 {W_{lh}}^2(l,h)}{2\sigma^2 W_{ll}(l,h)} \\ &- (\log W_l(l,h) + 1) = 0, \quad W \underset{t\to\infty}{\longrightarrow} 0. \end{aligned} \tag{7.21}$$

We see that in this case the value function $V(l,h,t) = e^{-\kappa t} W(l,h)$ and the complete dependence on t is described just by the factor $e^{-\kappa t}$. Here the investment $\pi(l,h)$ and consumption $c(l,h)$ look as follows in terms of the function $W(l,h)$

$$\pi(l,h) = -\frac{\eta\rho\sigma h W_{lh}(l,h) + (\alpha - r) W_l(l,h)}{\sigma^2 W_{ll}(t,l,h)}, \tag{7.22}$$

$$c(l,h) = h\frac{\kappa}{W_l(l,h)}. \tag{7.23}$$

To find the invariants of H_4 when $\omega \neq r/\kappa$ we can move according to a standard procedure. We obtain three independent invariants using a corresponding characteristic system

$$inv_1 = z = \frac{l}{h}, \quad inv_2 = \tau = (r - \kappa\omega)t - \log h, \tag{7.24}$$

$$inv_3 = W(z,\tau) = e^{\kappa t} V + \left(\omega - \frac{r}{\kappa}\right) t. \tag{7.25}$$

Analogously substituting expressions for the invariants z and τ (7.24) as the new independent and $W(z,\tau)$ (7.25) as the new dependent variables into (5.35) we get

$$\begin{aligned}
& (r-\kappa\omega)W_\tau(z,\tau) - \kappa W(z,\tau) + \frac{1}{2}\eta^2\left(2zW_z(z,\tau)+z^2W_{zz}(z,\tau)\right) \\
+ \;& (rz+\delta)W_z(z,\tau) - (\mu-\delta)zW_z(z,\tau) \qquad (7.26)\\
- \;& \frac{(\alpha-r)^2W_z^2(z,\tau) - 2(\alpha-r)\eta\rho W_z(z,\tau)(W_z(z,\tau)+zW_{zz}(z,\tau))}{2\sigma^2W_{zz}(z,\tau)} \\
- \;& \frac{\eta^2\rho^2\sigma^2(W_z(z,\tau)+zW_{zz}(z,\tau))^2}{2\sigma^2W_{zz}(z,\tau)} \\
- \;& \log W_z(z,\tau) - \tau - \omega + \frac{r}{\kappa} - 1 = 0, \quad W \underset{t\to\infty}{\longrightarrow} 0. \qquad (7.27)
\end{aligned}$$

Here the investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ look as follows in terms of the value function W

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z + \frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{W_z(z,\tau)}{W_{zz}(z,\tau)}\right), \qquad (7.28)$$

$$c(z,\tau,h) = h\frac{\kappa}{W_z(z,\tau)}. \qquad (7.29)$$

Case $H_5(h_5)$. This subalgebra h_5 is defines as

$$h_5 = < e_1 \pm e_4 > = \left\langle \left(\frac{r}{\kappa}l \pm e^{rt}\right)\frac{\partial}{\partial l} + \frac{r}{\kappa}h\frac{\partial}{\partial h} + \frac{1}{\kappa}\frac{\partial}{\partial t} - \left(V - \frac{r}{\kappa^2}e^{-\kappa t}\right)\frac{\partial}{\partial V}\right\rangle.$$

According to a standard procedure for finding the invariants of the subgroup H_5 we obtain three independent invariants as a solution of the characteristic system

$$inv_1 = x = le^{-rt} \mp \kappa t, \quad inv_2 = y = he^{-rt}, \qquad (7.30)$$

$$inv_3 = W(x,y) = e^{\kappa t}V + \frac{r}{\kappa}t. \qquad (7.31)$$

Substituting the new independent variables x, y (7.30) and the new dependent variable $W(x,y)$ (7.31) into (7.11) we get a two dimensional PDE

$$\begin{aligned}
\pm \;& \kappa W_x(x,y) - \kappa W(x,y) + \frac{1}{2}\eta^2\tau^2W_{yy}(x,y) + \delta yW_x(x,y) + (\mu-\delta)yW_y(x,y) \\
- \;& \frac{(\alpha-r)^2W_x^2(x,y) + 2(\alpha-r)\eta\rho yW_x(x,y)W_{xy}(x,y) + \eta^2\rho^2\sigma^2y^2{W_{xy}}^2(x,y)}{2\sigma^2W_{xx}(x,y)} \\
- \;& \log W_x(x,y) + \frac{r}{\kappa} - 1 = 0, \quad W(x,0) \underset{x\to\mp\infty}{\longrightarrow} 0. \qquad (7.32)
\end{aligned}$$

Here the investment $\pi(x,y,h)$ and consumption $c(x,y,h)$ look as follows in terms of the value function $W(x,y)$

$$\pi(x,y,h) = -h\frac{\eta\rho\sigma W_{xy}(x,y) + (\alpha-r)y^{-1}W_x(x,y)}{\sigma^2 W_{xx}(x,y)}, \tag{7.33}$$

$$c(x,y,h) = h\frac{1}{W_x(x,y)}. \tag{7.34}$$

Case $H_7(h_7)$. The last one dimensional subalgebra in the list of optimal system of subalgebras in Table 7.2 is defined as

$$h_7 =< e_2 \pm e_3 >= \left\langle l\frac{\partial}{\partial l} + h\frac{\partial}{\partial h} + \left(\frac{1}{\kappa}e^{-\kappa t} \mp 1\right)\frac{\partial}{\partial V}\right\rangle.$$

According to a standard procedure we look for invariants of the subgroup H_7 and obtain three independent invariants

$$inv_1 = t, \quad inv_2 = z = \frac{l}{h}, \tag{7.35}$$

$$inv_3 = W(z,t) = V - \left(\frac{1}{\kappa}e^{-\kappa t} \mp 1\right)\log h \tag{7.36}$$

Using the invariants (7.35) as the new independent variables z,t and the invariant (7.36) as the new dependent variable $W(z,t)$ and substituting them into (5.35) we obtain a two dimensional PDE

$$\begin{aligned} W_t(z,t) &+ \frac{1}{2}\eta^2\left(2zW_z(z,t) + z^2W_{zz}(z,t)\right) + (rz+\delta)W_z(z,t) - (\mu-\delta)zW_z(z,t) \\ &- \frac{(\alpha-r)^2W_z^2(z,t) - 2(\alpha-r)\eta\rho W_z(z,t)(W_z(z,t) + zW_{zz}(z,t))}{2\sigma^2 W_{zz}(z,t)} \\ &- \frac{\eta^2\rho^2\sigma^2(W_z(z,t) - zW_{zz}(z,t))^2}{2\sigma^2W_{zz}(z,t)} \\ &- e^{-\kappa t}\left(\log W_z(z,t) - \frac{1}{2}\eta^2 - \mu + \delta\right) = 0. \end{aligned} \tag{7.37}$$

We list this reduction here, but need to note that this substitution means that condition $V(t,l,h) \underset{t\to\infty}{\longrightarrow} 0$ is not satisfied. So, the solution of (7.37) would not be a solution of the original problem, that is why we do not demonstrate how $\pi(z,t,h)$ and $c(z,t,h)$ look in this case.

7.2.3 Two dimensional subalgebras of L_4^{LOG} and corresponding reductions

With the help of Table 7.2 we can find the deeper reductions that can reduce PDE (7.11) to an ODE. To do that we have to look on two parameter subalgebras listed in the second row of Table 7.2.

Case $H_8(h_8)$. The first two dimensional subalgebra listed in Table 7.2 is subalgebra $h_8 =< e_1, e_3 >$ spanned by two generator defined as follows

$$\begin{aligned} e_1 &= \frac{r}{\kappa} l \frac{\partial}{\partial l} + \frac{r}{\kappa} h \frac{\partial}{\partial h} + \frac{1}{\kappa} \frac{\partial}{\partial t} - \left(V - \frac{r}{\kappa^2} e^{-\kappa t} \right) \frac{\partial}{\partial V}, \\ e_3 &= -l \frac{\partial}{\partial l} - h \frac{\partial}{\partial h} - \frac{1}{\kappa} e^{-\kappa t} \frac{\partial}{\partial V}. \end{aligned}$$

Since we have studied before both of these generators, we can use our previous results and rewrite one of the equations in the characteristic systems in terms of invariant variables of the other one.

Indeed if we assume that $\omega = 0$ in the case $h_4 =< e_1 + \omega e_3 >$ then the invariants listed in (7.24) will be invariants of $< e_1 >$, i.e. we get

$$\begin{aligned} inv_1 &= z = \frac{l}{h}, \quad inv_2 = \tau = rt - \log h, \\ inv_3 &= W(z, \tau) = e^{\kappa t} V - \frac{r}{\kappa} t. \end{aligned}$$

If we rewrite the second generator e_3 of subalgebra h_8 in terms of these three invariants z, τ and W as new independent and dependent variables correspondingly, we obtain

$$e_3 = \frac{\partial}{\partial \tau} - \frac{1}{\kappa} \frac{\partial}{\partial W}. \tag{7.38}$$

Solving a corresponding characteristic system $\frac{d\tau}{1} = \frac{dW}{-1/\kappa}$ we obtain a new common invariant

$$inv_{e_3} = Y(z) = \kappa W(z, \tau) + \tau. \tag{7.39}$$

This is a solution invariant under the action of two parameter subgroup H_8. Now we can take $Y(z)$ as a new dependent variable in (7.11) and z as a new independent one. Substituting these invariants into PDE (6.18) with $\omega = 0$ we obtain a new ODE

$$\begin{aligned}
& Y(z)+\frac{1}{2}\eta^2\left(2zY_z(z)+z^2Y_{zz}(z)\right)+(rz+\delta)Y_z(z)-(\mu-\delta)zY_z(z)\\
- & \frac{(\alpha-r)^2Y_z^2(z)-2(\alpha-r)\eta\rho Y_z(z)(Y_z(z)+zY_{zz}(z))+\eta^2\rho^2\sigma^2(Y_z(z)-zY_{zz}(z))^2}{2\sigma^2Y_{zz}(z)}\\
- & \log Y_z(z)+\log\kappa-1=0. \qquad (7.40)
\end{aligned}$$

The investment $\pi(z,\tau,h)$ and consumption $c(z,\tau,h)$ in terms of $Y(z)$ now look like

$$\pi(z,\tau,h) = h\left(\frac{\eta\rho}{\sigma}z+\frac{\eta\rho\sigma-\alpha+r}{\sigma^2}\frac{Y_z(z)}{Y_{zz}(z)}\right), \qquad (7.41)$$

$$c(z,\tau,h) = h\frac{\kappa^2}{Y_z(z)}. \qquad (7.42)$$

In terms of original variables t,l,h and $V(t,l,h)$ the substitution looks as follows

$$\begin{aligned}
z &= \frac{l}{h}, \quad \tau=rt-\log h, \qquad (7.43)\\
Y(z) &= \kappa e^{\kappa t}V(t,l,h)-\log h.
\end{aligned}$$

It also means that if we obtain a solution $Y(z)$ for (7.40) we obtain the value function that in terms of original variables looks like

$$V(t,l,h)=\frac{e^{-\kappa t}}{\kappa}Y(l/h)+\frac{e^{-\kappa t}}{\kappa}\log h,$$

and the condition $V(l,h,t)\underset{t\to\infty}{\longrightarrow}0$ is satisfied.

All other two dimensional subalgebras listed in Table 7.2 do not give any meaningful reductions of the original equation (7.11) to an ODE, so we will not regard them in detail.

7.3 Results of the chapter

In Chapter 5 the Lie group analysis of the problem was carried out. In this chapter we have found all Lie type reductions of the equations with logarithmic utility function that follow from the results of that analysis.

Using the notation provided in [50] we show that for a general liquidation time distribution L_3^{LOG} can be classified as $A_1\bigoplus A_2$. In a special case of exponentially distributed liquidation function the admitted Lie algebra is four dimensional is classified as $A_2 \oplus A_2$.

For all admitted algebras we used a system of optimal subalgebras. We provide all non equivalent reductions of the studied equations and describe the solutions which can not be transformed to each other with a help of the transformations from the admitted symmetry group. We list all possible Lie type reductions of the problem for every case and demonstrate how the corresponding optimal policies look like in a feedback form.

Once again, as well as in HARA utility case, the optimal policies tend to classical Merton policies as $h \to 0$, which is only to be expected, since by design the model, when $h = 0$ corresponds to a portfolio without an illiquid asset.

The results of Chapter 6 and Chapter 7 are especially important since there are several models connected with the optimization of a portfolio in a presence of a random income and due to the growing importance of liquidity as a financial factor that affects market behavior one can expect further developments in this area. We have mentioned several works that use some of the substitution mentioned in these chapters without any explanation of this usage. It is only to be expected that new related models will emerge and we hope that listed substitutions and the proved possibility to reduce the HJB equation to an ODE in the case of exponentially distributed liquidation time can help the other researchers in the field to broaden their scope and regard simplified equations when need be or change their model setting in correspondence with their research focus.

In the next, and final, chapter we summarize the results of this work.

8

Conclusion

In this work we regard a portfolio optimization problem for a basket consisting of a riskless liquid, risky liquid and risky illiquid assets. The illiquid asset is sold in a random moment of time T that has a distribution with a survival function $\overline{\Phi}(t)$, satisfying general conditions $\lim_{t\to\infty} \overline{\Phi}(t)E[U(c(t))] = 0$ and $\overline{\Phi}(t) \sim e^{-\kappa t}$ or faster as $t \to \infty$. This, to our knowledge, is a new generalization of the illiquidity models that opens a new class of optimization problems within the adapted resource allocation approach developed in [41]. At the same time this model has considerable importance for the practical needs of financial markets, since the importance of the mathematically tractable models of illiquidity was understood after the global financial crisis of 2009 and draws constant attention of the practitioners.

In Chapter 2 we explain economical meaning and significance of financial liquidity. We list the most important approaches to the issue of liquidity starting with purely qualitative definitions and finishing with quantitive models. We discuss the advantages and disadvantages of every model and pay special attention to the results in the are of portfolio optimization, since we believe that this approach is the most interesting and promising.

In Chapter 3 we formulate the problem in a framework that was proposed in [55] but make it considerably broader, showing how one can regard an exogenous random liquidation time instead of a fixed deterministic time horizon. We also show how one can move from this problem with random time horizon to a problem with infinite horizon and special weight function that is characterized by the probability distribution of the liquidation time T. Further in this work we focus on the latter formulation of the problem. Applying the technique of so-called viscosity solutions to the problem we prove the existence and uniqueness of the solution for a broad class of liquidation time distributions. It is important to note here, that though particular results similar to the ones described in this work were obtained before for different problems, this is a first result of such generality obtained in the context of illiquidity models.

In Chapter 4 we consider specific liquidation time distributions, namely, exponential and Weibull distributions, in order to show that the formulation of the problem proposed in Chapter 3 is actually applicable to the real world and provides optimal strategies that differ from the policies, found by Merton in [42], yet that policies tend to the classical Merton ones as the share of illiquid capital vanishes. This way we show that our model stands in line with the optimization models that did not take liquidity into consideration, but at the same time it gives new results in presence of liquidity factor. We work with exponential distribution, since it is widely regarded in the literature and, indeed, has certain distinguishing properties. As an example of another distribution we use Weibull, since it is a distribution that is often regarded in the context of illiquid assets. For both distributions we find upper and lower bounds and compare these two cases using a numerical simulation.

In Chapter 5 we carry out a complete Lie group analysis for two different utility functions, namely, HARA and logarithmic utility functions that correspond to two different three dimensional PDEs (5.20) and (5.35) with an arbitrary function $\overline{\Phi}(t)$. We solve these voluminous problems and find the admitted Lie algebras L_3^{HARA} and L_3^{LOG} correspondingly. The study of such three dimensional problems is, unfortunately, connected with a lot of 'manual' calculations. This is especially relevant when one has to work with an equation that has an arbitrary function. We obtain two voluminous systems of partial differential equations (137 equations in the first system and 130 equations in the other) that define the generators of the corresponding algebras. Solving these systems is a step-by-step handmade procedure that can be only slightly facilitated with certain computer packages. As far as we know, this is a first Lie group analysis of such problem for a general liquidation time distribution.

However, we demonstrate that there is a possibility to choose HARA-utility in such a form that $U^{HARA} \underset{\gamma \to 0}{\longrightarrow} U^{LOG}$. This fact was mentioned in some publications before but, to our knowledge, it was not demonstrated explicitly, in a step-by-step manner how deep this connection is on the analytical level. May be because of that majority of the HARA utilities that we found in the literature do not actually posses this quality. They are certainly utility functions of the HARA type but the limit procedure applied to such utility function can at its best be reduced to some modification of logarithmic utility (not $\log c$), while some of them do not even have anything that remotely resembles a logarithm. To demonstrate the connection between two problems we choose U^{HARA} in the form (5.18) and show that not only we obtain correct form of logarithmic utility as $\gamma \to 0$, i.e. $U^{HARA} \underset{\gamma \to 0}{\longrightarrow} U^{LOG}$ but, naturally, a three-dimensional HJB equation (3.11) corresponding to HARA-utility formally transforms into an HJB-equation that corresponds to logarithmic case as $\gamma \to 0$. After a formal maximization of (3.11) we obtain three dimensional PDEs corresponding to HARA and logarithmic utility correspondingly. We demonstrate that the PDE (5.35) arising in the case of logarithmic utility function can be formally regarded as a limit case of the PDE (5.20) arising in the case of HARA utility function as $\gamma \to 0$.

We also show that if and only if the liquidation time defined by a survival function $\overline{\Phi}(t)$ is distributed exponentially, then for both types of the utility functions we get an additional symmetry. We prove that both Lie algebras admit this extension, i.e. we obtain the four dimensional L_4^{HARA} and L_4^{LOG} correspondingly for the case of exponentially distributed liquidation time. Indeed, the case of exponentially distributed liquidation time is actually similar to the infinite-horizon random income problem and several other models studied in the literature (see, for instance, [22]), yet our work is the first to our knowledge that explicitly shows which properties make an exponentially distributed liquidation time a distinguished case, that allows a reduction of an original three-dimensional PDEs to ODEs. This result is particularly important, since there is a number of works in the field that use similar reductions without any explanation that there is actually no other distribution that allows a Lie type reduction of a PDE to an ODE. With a help of Lie group analysis we explain what makes exponential liquidation time distribution a distinguished case, the only situation when one can reduce the three dimensional HJBs to ODEs.

In Chapter 6 and Chapter 7 we study the internal structure of the admitted Lie algebras further in order to use their structures and obtain convenient and useful reductions of both PDEs (5.20) and (5.35), correspondingly. Using the notation provided in [50] we show that for a general liquidation time distribution L_3^{HARA} can be classified as $A_{3,5}^{\gamma}$ and L_3^{LOG} as $A_1 \bigoplus A_2$. We use the system of optimal subalgebras provided in [50] and obtain corresponding reductions of both three dimensional PDEs. We show how each of the original problems can be reduced to a corresponding two dimensional one and prove that in general case with an arbitrary function $\overline{\Phi}(t)$ there is no Lie type reduction that leads to an ODE.

We also look at the symmetry algebras L_4^{HARA} and L_4^{LOG} admitted by the corresponding equations (6.11) and (7.11) in the case of exponential liquidation time distribution and either HARA or logarithmic utility respectively. In general, L_4^{HARA} can be classified as $A_2 \oplus A_2$ in terms of the notation used in [50]. However, if $\kappa = r\gamma$ the symmetry algebra L_4^{HARA} has the structure that corresponds to $A_{3,5}^{\gamma} \bigoplus A_1$, in the notation of [50]. L_4^{LOG} is classified as $A_2 \oplus A_2$.

Using a system of optimal subalgebras for all admitted algebras allows us to provide all non equivalent reductions of the studied equations and describe the solutions which can not be transformed to each other with a help of the transformations from the admitted symmetry group. For every case we list all possible Lie type reductions of the problem. The reduced equations that are two dimensional PDEs or in some special cases are even ODEs. Such equations are much more convenient for further analytical or numerical studies. We also show how one can rewrite corresponding optimal policies in every case.

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Declaration

I herewith declare that I have produced this paper without the prohibited assistance of third parties and without making use of aids other than those specified; notions taken over directly or indirectly from other sources have been identified as such. This paper has not previously been presented in identical or similar form to any other German or foreign examination board.

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Erklärung zur selbständigen Anfertigung der Dissertation

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne fremde Hilfe verfasst, die Zitate ordnungsgemäßgekennzeichnet und keine anderen als die im Literaturverzeichnis angegebenen Quellen und Hilfsmittel benutzt habe.

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